RESEARCH STATEMENT

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Introduction

My primary research interests lie in complex analysis (in one variable), especially in complex-valued analytic function spaces and their applications in operator theory and potential theory. In my doctoral research, I work on the following four problems on Bergman spaces $\mathcal{A}^2(\mathbb{D})$ defined on the unit disk $\mathbb{D}$ [6].

(1) Given $0 < c < 1$, find $F(c) := \inf_{f,g \in S^1} \sup_{c \le |z| < 1} |f(z)/g(z)|$ where $S^1 = \{f \in \mathcal{A}^2(\mathbb{D}) : \ \|f\|_2 = 1\}$ denotes the unit ball in Bergman space $\mathcal{A}^2(\mathbb{D})$ and $\|f\|_2 := (\int_{\mathbb{D}} |f(z)|^2 \, dA(z))^{1/2}$.

(2) Given $0 < c < 1$, find the extremal pair of functions $f_0, g_0 \in S^1$ such that $F(c) = |f_0(z)/g_0(z)|$.

(3) Given $0 < c < 1$, $F_B(c) := \sup_{f,g \in FG(c)} \left(\|f\|_2^2 - \|g\|_2^2\right)$, where $FG(c) := \{f, g \in \mathcal{A}^2(\mathbb{D}) : \|f\|_2 \le 1, \ \|g\|_2 \le 1, \ |f(z)| \le |g(z)|, \ \forall z : c \le |z| < 1\}$ is the set of all admissible pairs for $F_B(c)$.

(4) Given $0 < c < 1$, find the extremal pair of functions $f_1, g_1 \in FG(c)$ such that $F_B(c) = \|f_1\|_2^2 - \|g_1\|_2^2$.

The inspiration behind these problems is *Korenblum's maximum principle in Bergman spaces* [3], which is an analog of the classical maximum principle for Bergman spaces: *For $f, g \in \mathcal{A}^2(\mathbb{D})$, there is a constant $0 < c < 1$, such that if $|f(z)| \le |g(z)|$ for all z with $c < |z| < 1$, then $\|f\|_2 \le \|g\|_2$.* Korenblum's maximum principle can be thought of as a version of the Comparison Theorems [15] which are well known from the theory of partial differential equations. Let us call the largest possible value of such c, denoted by κ, as the *Korenblum's constant.* W. Hayman [9] proved the existence of κ, however the exact value of κ remains unknown. Several partial results can be found in [10, 13, 14, 16, 18, 19]. It is easy to note that $0 < F(c) \le 1$ and in fact, $F(c)$ is a non-increasing function in $(0, 1)$. Therefore $F(c) = 1$ for $0 < c \le \kappa$ and $F(c) < 1$ for $\kappa < c < 1$. Similarly, we observe that $0 \le F_B(c) < 1$ and $F_B(c)$ is a non-decreasing function. Therefore, $F_B(c) = 0$ for $0 < c \le \kappa$ and $F_B(c) < 1$ for $\kappa < c < 1$. Thus, on a brighter note, the complete solution of the above mentioned four problems would solve Korenblum's maximum principle. So far, I have proved the following six results:

(A) The extremal pair of functions exists for problems (1) and (3) and they are bounded.

(B) Both $F(c)$ and $F_B(c)$ are monotone on $(0, 1)$ and strictly monotone on $(\kappa, 1)$.

(C) Both F, $F_B : [\kappa, 1] \to [0, 1]$ are homeomorphisms.

(D) $F(c) > \sqrt{1 - c^2}$ and $F_B(c) \le c^2$ for $0 < c < 1$

(E) $\kappa = \lim_{n\to\infty} \kappa_n$ where κ_n is the Korenblum's constant for the class of polynomials of degree at most n, where $n \ge 1$.

(F) For the class of degree ≤ 1 polynomials, let $0 < c < 1$, $f(z) = \frac{\alpha+\beta z}{\sqrt{\alpha^2+\frac{\beta^2}{2}}}$, $g(z) = \frac{\gamma+\delta z}{\sqrt{\gamma^2+\frac{\delta^2}{2}}}$, where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. Then $F(c) = 1$ for $0 \le c \le \frac{1}{\sqrt{2}}$ and $F(c) = \frac{1}{\sqrt{2}c}$ for $\frac{1}{\sqrt{2}} < c \le 1$. In particular, $\kappa_1 = \frac{1}{\sqrt{2}}$ and the extremal polynomials are $f(z) = 1$, $g(z) = \sqrt{2}z$.

I am currently working to prove that the extremal pair of functions is unique for every c in $(\kappa, 1)$ up to rotation. I suspect that the extremal pair of functions for the general class of functions for both problems (1) and (3) are *polynomials*. I am also working to find the Korenblum's constant κ_2 for the class of polynomials of degree at most two. These are to be included in my Ph.D. thesis which is to be submitted in Spring 2016.

I plan to study the problems (1)–(4) in Bergman spaces $\mathcal{A}^p(\mathbb{D})$ for $p \geq 1$ [6] and investigate whether there is any relationship between the extremal pair of functions in $\mathcal{A}^p(\mathbb{D})$ and $\mathcal{A}^q(\mathbb{D})$ for $1 < p, q < \infty$ and $1/p + 1/q = 1$. This will in turn suffice to find a solution in $\mathcal{A}^p(\mathbb{D})$ in order to get a solution for $\mathcal{A}^q(\mathbb{D})$ space. I also plan to investigate whether $\{\kappa_p\}_{p\geq 1}$ is (eventually) monotonic as a function of p, where κ_p is the Korenblum's constant for $\mathcal{A}^p(\mathbb{D})$ space.

I have also developed an interest in a problem conjectured by A. Solynin [17] related to the growth process of the hyperbolic polygons in the unit disk with all its vertices on the unit circle and each n-gon generates an $n(n-1)$-gon by reflecting itself across all of its own sides. I proved that [4] "*Among all hyperbolic* 4*-gons with center at the origin, the inverted side length of the longest side is minimal for the regular hyperbolic* 4*-gon.*"

Background

An annulus in the complex plane $\mathbb{C}$ is defined as $A(r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z| < r_2\}$. The classical Maximum Modulus Theorem [5] states that if a function f is analytic in the unit disk $\mathbb{D}$ and $|f(z)| \leq M$ in $A(c, 1)$ for some constants M and c $(M, c > 0)$, then $|f(z)| \leq M$ for all z in $\mathbb{D}$. Therefore, $\|f\|_2 \leq M = \|M\|_2$. It is quite natural to ask what happens when M is replaced by any arbitrary non-constant analytic function on $\mathbb{D}$. On this note, Boris Korenblum [12] conjectured in 1991 that "*for* $f, g \in \mathcal{A}^2(\mathbb{D})$*, there is a constant* $0 < c < 1$ *such that if* $|f(z)| \leq |g(z)|$ *for all* $z \in A(c, 1)$ *then* $\|f\|_2 \leq \|g\|_2$" where the Bergman space $\mathcal{A}^2(\mathbb{D})$ is the class of analytic functions f on $\mathbb{D}$ with $\|f\|_2 := \left(\frac{1}{\pi}\int_{\mathbb{D}} |f(z)|^2\, dA(z)\right)^{1/2} < \infty$, where $dA = r\, dr\, d\theta$ denotes the Lebesgue area measure. Korenblum [12] proved a weaker version of this conjecture with an additional assumption that each zero of f is a zero of g. It is easy to observe that if the quotient $\frac{f}{g}$ is analytic then by the Maximum Modulus Theorem, $|f(z)| \leq |g(z)|$ in $\mathbb{D}$ which further implies that $\|f\|_2 \leq \|g\|_2$. However, using a simple example, Hayman [9] in 1999 showed that this conclusion is not true in general if we replace $\mathbb{D}$ by $A(c, 1)$ and he also proved the conjecture for $c = 0.04$. Therefore, this conjecture is sometimes referred as *Korenblum's maximum principle* or *the Bergman space maximum principle*. The immediate next question which interested many researchers is, "*What is the best possible or sharp value (say* κ*) of* c *for which Korenblum's conjecture will be true for all pair of functions* (f, g) *in* $\mathcal{A}^2(\mathbb{D})$?". Although the exact value of κ is still unknown, various partial results in connection to finding bounds of κ were offered in a series of papers by Korenblum, Richards, O'Neil, Zhu [13], Hinkkanen [9], Schuster [16], Wang [18, 19] and many others. Additionally, Hinkkanen and Wang generalized their results in $\mathcal{A}^p(\mathbb{D})$ for $p \geq 1$. In some recent papers of Wang [18, 19], the best known bounds of κ can be found as $0.28185 < \kappa < 0.6778994$. Furthermore, Hayman and Danikas [8] studied Korenblum-type problems in Hardy spaces and the space of disk algebras and Schuster studied this problem [16] in Fock spaces.

My current work revolves around the extremal problems (1)–(4) for Bergman spaces. Since Bergman spaces can be interpreted as an extension of Hardy spaces, analogous problems in Hardy spaces are studied in Bergman spaces. Extremal problems in Bergman spaces have many applications in diverse areas such as in potential theory, operator theory and Dirichlet spaces (which are related to PDEs with solutions as p-harmonic functions). Extremal problems in Hardy and Dirichlet spaces were thoroughly studied by S. Ya Khavinson, Rogosinki and many others. The general extremal problems have been extensively studied for many years since it is not obvious what extremal functions satisfy the problem. Unlike Hardy spaces, the standard techniques of functional analysis fail for solving linear extremal problems in Bergman spaces. There have been numerous attempts to develop a theory of Hahn-Banach dual linear extremal problems for Bergman spaces with partial discoveries obtained in [20]. An excellent survey of linear extremal problems can be found in [2]. Therefore, as there are many basic open questions, the theory in Bergman spaces is still in its infancy.

Completed Research

I have studied the properties of the functions $F(c)$ and $F_B(c)$, that I refer as *Korenblum's function for problem* (1) and (3) respectively. It is easy to notice that $0 < F(c) \leq 1$ and in fact, $F(c)$ is a non-increasing function in $(0, 1)$. Therefore $F(c) = 1$ for $0 < c \leq \kappa$ and $F(c) < 1$ for $\kappa < c < 1$. Similarly, we observe that $0 \leq F_B(c) < 1$ and $F_B(c)$ is a non-decreasing function. Therefore, $F_B(c) = 0$ for $0 < c \leq \kappa$ and $F_B(c) < 1$ for $\kappa < c < 1$. I proved the following five theorems which extend the knowledge and understanding of the behavior of the functions $F(c)$ and $F_B(c)$.

Theorem 1. *Let* $F : (0, 1) \to (0, 1)$ *and* $F_B : (0, 1) \to (0, 1)$ *be as defined before.*

(a) *Both* $F(c)$ *and* $F_B(c)$ *are strictly monotone on* $(\kappa, 1)$.

(b) *Both* F, $F_B : [\kappa, 1] \to [0, 1]$ *are homeomorphisms.*

I proved Theorem 1 using standard, though, technically involved analytical arguments. The following theorem provides a lower bound for $F(c)$ and an upper bound for $F_B(c)$.

Theorem 2.

(a) $F_B(c) \leq c^2$ *for* $0 < c < 1$.

(b) $F(c) > \sqrt{1 - c^2}$ *for* $0 < c < 1$.

Proof of Theorem 2(a) was done by the fact that $F_B'(c)$ exists almost everywhere that is guaranteed by Theorem 1. I selected one such point c and assumed that (f, g) is an extremal pair of functions for $F_B(c)$. Further the dilation of the extremal functions f and g as $f_r(z) := f(rz)$ and $g_r(z) := g(rz)$ respectively, where r is sufficiently close to 1, played a key role in finding an inequality between the quotient $\frac{F_B'(c)}{F_B(c)}$ and an explicit function of c that completed the proof. To prove Theorem 2(b), it can be noticed that $(f_1, F(c)g_1)$ is admissible for $F_B(c)$ if (f_1, g_1) is an extremal pair of functions for $F(c)$ which allowed me to use Theorem 2(a) to reach the conclusion.

Although it is difficult to work with the general problems (1)–(4), we find the extremal pair of functions for the class of linear polynomials as a representative case. Hayman [9] considered the example $f(z) = a$, $g(z) = z$ with

$\frac{1}{\sqrt{2}} < a < c$ to show that the Korenblum's conjecture fails for $c > \frac{1}{\sqrt{2}}$. We define κ_n to be the *Korenblum's constant for* $\mathcal{P}_n$ where $\mathcal{P}_n$ denotes the class of polynomials of degree at most n, where $n \geq 1$. The following theorems 3 and 4 give the solutions of problems (1)–(4) and guarantee that $\frac{1}{\sqrt{2}}$ is indeed the sharp constant for the same representative case of the class of linear polynomials!

Theorem 3. *Given* c, $0 < c < 1$ *and* $f(z) = \dfrac{\alpha + \beta z}{\sqrt{\alpha^2 + \frac{\beta^2}{2}}}$, $g(z) = \dfrac{\gamma + \delta z}{\sqrt{\gamma^2 + \frac{\delta^2}{2}}}$, *the Korenblum's function for the class of linear polynomials for problem (1) is given by*

$$F(c) = \begin{cases} 1 & \text{if} \quad 0 \leq c \leq \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}c} & \text{if} \quad \frac{1}{\sqrt{2}} < c \leq 1. \end{cases}$$

In particular, Korenblum's constant for linear polynomials is $\kappa_1 = \frac{1}{\sqrt{2}}$ *and the extremal polynomials are* $f(z) = 1$, $g(z) = \sqrt{2}z$.

An immediate observation of the forms of extremal polynomials in the linear case suggests that $f(z) = 1$ and $g(z) = \sqrt{n+1}z^n$ may provide better estimates for the case of $n \geq 2$. However, a straightforward set of calculations [3] confirms that these pair of polynomials are not extremal. The proof of Theorem 3 is based on the fact that the quotient $\frac{\alpha+\beta z}{\gamma+\delta z}$ is a Möbius map and hence its properties can be exploited to reach the conclusion. Using a similar method as in Theorem 3, it is also possible to find the extremal pair of functions for a special case of $\mathcal{P}_2$ where functions are of the forms $f(z) = \alpha + \beta z$ and $g(z) = z(\gamma + \delta z)$. In this case, the Korenblum's constant is again $\frac{1}{\sqrt{2}}$. The following result shows that using the definition of $F_B(c)$ from problem (2), the extremal pair of functions for linear polynomials depends on c and is different from what we found in Theorem 3. However as expected, Korenblum's constant is again $\frac{1}{\sqrt{2}}$.

Theorem 4. *Given* $0 < c < 1$, $f(z) = \alpha + \beta z$ *and* $g(z) = \gamma + \delta z$, *the Korenblum's function for the class of linear polynomials for problem (1) is given by,*

$$F_B(c) := \max\left(1 - |\gamma|^2 - \frac{|\delta|^2}{2}\right) = \begin{cases} 0\,, & \text{if} \quad 0 \leq c \leq \frac{1}{\sqrt{2}} \\ 1 - \frac{1}{2c^2}\,, & \text{if} \quad \frac{1}{\sqrt{2}} < c \leq 1 \end{cases}$$

where the maximum is taken over all α, β, γ, $\delta \in \mathbb{C}$ *such that* $|\gamma|^2 + |\delta|^2/2 \leq 1$, $|\alpha|^2 + |\beta|^2/2 \leq 1$ *and* $|f(z)| \leq |g(z)|$ *for all* z *with* $c \leq |z| < 1$ *i.e.,* $|\alpha + \beta re^{i\theta}| \leq |\gamma + \delta re^{i\theta}|$ *for all* $0 \leq \theta \leq 2\pi$ *and for all* $c \leq r \leq 1$. *In particular, Korenblum's constant for linear polynomials is* $\kappa_1 = \frac{1}{\sqrt{2}}$ *and the extremal polynomials are* $f(z) = 1$, $g(z) = \frac{z}{c}$.

I am also revisiting the crucial question related to uniqueness of the extremal pair of functions which remains unanswered, "*Is an extremal pair of functions* (f, g) *for* $F(c)$ *unique for every* c *in* $(\kappa, 1)$ *up to a rotation*?" Theorems 3 and 4 assure that the extremal pair of functions is indeed unique for the class of linear polynomials. However, due to the complex nature of the non-linear functionals, the uniqueness is not easily tractable for the general class of polynomials $\mathcal{P}_n$ with $n \geq 2$.

We define κ^b and κ_n^B to be the Korenblum's constant for bounded functions and Blaschke products of order at most $n \geq 1$, respectively. The following results are quite interesting and potentially useful for the general problem.

Theorem 5.

(a) $\kappa = \lim_{n\to\infty} \kappa_n = \lim_{n\to\infty} \kappa_n^B$

(b) $\kappa = \kappa^b$.

I proved Theorem 5(a) using the fact that any analytic function has a power series expansion in $\mathbb{D}$ and therefore can be approximated by a sequence of partial sums. Theorem 5(b) guarantees that the extremal pair of functions are indeed bounded for a wider class of functions in $\mathcal{A}^2(\mathbb{D})$. I proved Theorem 5(b) by using the fact that any analytic function f defined in $\mathbb{D}$ can be approximated by the sequence of dilations f_r where r is sufficiently close to 1. To prove the case for Blaschke products in Theorem 5(a), I used Theorem 5(b) and the fact that a bounded function f (with norm $\sup_{z\in\mathbb{D}} |f(z)|$ at most 1) can be approximated by using a sequence of Blaschke products in $\mathbb{D}$.

Ongoing Research

I am currently working on the class $\mathcal{P}_2$ to answer problems (1)-(4) to compute κ_2. Since $\mathcal{P}_n \subseteq \mathcal{P}_{n+1}$ for $n \geq 1$, it follows that $0 < \kappa \leq \kappa_{n+1} \leq \kappa_n \leq \ldots \leq \kappa_2 \leq \kappa_1 < 1$. Additionally, I expect that κ_2 *will be strictly less than* κ_1 and hence will improve the upper bound of κ. In order to get an idea of the extremal pair of functions for $\mathcal{P}_2$, I am considering degree two polynomials with complex conjugate roots as $f(z) = (z - \rho e^{i\phi})(z - \rho e^{-i\phi})$ and $g(z) = (z - \gamma e^{it})(z - \gamma e^{-it})$. I am also investigating any possible relationship between an extremal pair of functions for problem (1) and that of problem (3) using variational arguments, which will produce a better picture of the extremal pair of functions. It would be also fascinating to see if there is a connection between the extremal pair of functions and their "canonical zero-divisors". It appears that the role of canonical zero divisors in Bergman spaces is an analog of what Blaschke products do in Hardy spaces. In fact, the discovery of canonical zero divisors or the theory of invariant subspaces (which was studied extensively by Hedenmalm [11]), was actually possible due to the study of linear extremal problems in Bergman spaces.

On a very different note from Korenblum's conjecture, I have developed an interest in a problem conjectured by A. Solynin [4]. He considered a growth process of hyperbolic polygons in the disk with all its vertices on the unit circle such that each n-gon generates an $n(n-1)$-gon by reflecting itself across all of its own sides. This growth process was suggested by J. Hersch. A. Solynin [17] proved that the minimal growth of the conformal radius under this process occurs when a polygon is regular. In connection with this work, A. Solynin further conjectured that "*Among all hyperbolic n-gons, the growth of the Euclidean area will be minimal for the regular hyperbolic n-gon*". Toward finding a complete solution to this conjecture, I am applying majorization technique [7]. I already proved that [3] "*Among all hyperbolic* 4*-gons with center at origin, the inverted side length of the longest side is minimal for the regular hyperbolic* 4*-gon.*" As a matter of fact, the sides of a hyperbolic polygon are parts of the circles which are tangent to each other. Therefore, the geometry in this problem may give a possible application in a relatively new area of mathematics such as the discrete conformal mapping and circle packing (a configuration of circles with specified patterns of tangency). Variational arguments in the theory of circle packing may be a promising line of attack to Solynin's conjecture.

Future Research

In the near future, I shall study the extremal problems (1)–(4) for $\mathcal{A}^p(\mathbb{D})$ for $p \geq 3$ and also for the weighted Bergman spaces. I am interested in investigating the behavior of the function $F_p(c)$ as $p \to \infty$. Since $\kappa_p \to 1$ as $p \to \infty$ (a known fact), I am interested in checking whether the sequence $\{\kappa_p\}_{p\geq 1}$ is (eventually) monotonic as a function of p. Moreover, I have also conjectured that if (f, g) is an extremal pair of functions for $F(c)$ with at least one common zero in the annulus $c \leq |z| < 1$, then $f \equiv g$.

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