

## **AC 2008-1325: INTERACTIVE INVOLUTE GEAR ANALYSIS AND TOOTH PROFILE GENERATION USING WORKING MODEL 2D**

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# Interactive Involute Gear Analysis and Tooth Profile Generation using Working Model 2D

## Abstract

Working Model 2D (WM 2D) is a powerful, easy to use planar multibody software that has been adopted by many instructors teaching *Statics*, *Dynamics*, *Mechanisms*, *Machine Design*, as well as by practicing engineers. Its programming and import-export capabilities facilitate simulating the motion of complex shape bodies subject to constraints. In this paper a number of WM 2D applications will be described that allow students to understand the basics properties of involute-gears and how they are manufactured. Other applications allow students to study the kinematics of planetary gears trains, which is known to be less intuitive than that of fix-axle transmissions.

## Introduction

There are numerous reports on the use of Working Model 2D in teaching Mechanical Engineering disciplines, including *Statics*, *Dynamics*, *Mechanisms*, *Vibrations*, *Controls* and *Machine Design*<sup>1-9</sup>.

Working Model 2D (WM 2D), currently available from Design Simulation Technologies<sup>10</sup>, is a planar multibody software, capable of performing kinematic and dynamic simulation of interconnected bodies subject to a variety of constraints. The versatility of the software is given by its geometry and data import/export capabilities, and scripting through formula and WM Basic language system. A number of textbooks<sup>11-15</sup> include simulation examples generated with WM 2D, and some even provide a student version of the software on companion CDs<sup>12,14</sup>.

In this paper several WM 2D applications will be described that allow students to understand the basics properties of involute gears, how they can be manufactured using rack or gear cutters, and what is the effect of addendum modification upon teeth geometry. Other WM 2D applications allow students to study the kinematics of two degree-of-freedom planetary gear trains, the motion of which is known to be less intuitive than that of fix-axle transmissions. In producing these applications, extensive use has been made of the interactive controls and scripting capabilities of the software.

## The Involute Curve of a Circle

Gear Theory is a specialized topic that undergraduate students in Mechanical Engineering are exposed to in *Kinematics and Dynamics of Machinery* and/or *Machine Design* classes.

It is known that involute gears are the most widely used in practice, being preferred over cycloidal and circular profile gears (like Wildhaber-Novikov), because of the following favorable properties<sup>16,17</sup>:

- the transmission ratio between two involute gears is not sensitive to the center distance modification;
- the same cutting tool can be used to manufacture gears with any number of teeth - the module (or diametral pitch) and whole depth of these gears will of course be the same;

- the cutting tools used to fabricate involute gears (in particular the rack and hob cutter), can be conveniently mass produced because of their cutting surfaces are straight and therefore easy to sharpen.

As their name suggests, involute gears have the active flanks of their teeth shaped as involute curves of a common circle, called *base circle*. The involute of a circle is obtained by attaching a taut, inextensible string to this given circle, and tracing its free end as it is wound or unwound onto the base circle<sup>18</sup>. Fig. 1 illustrates this concept, where BC represents the string, while the involute curve is the locus generated by the end point C of the string.

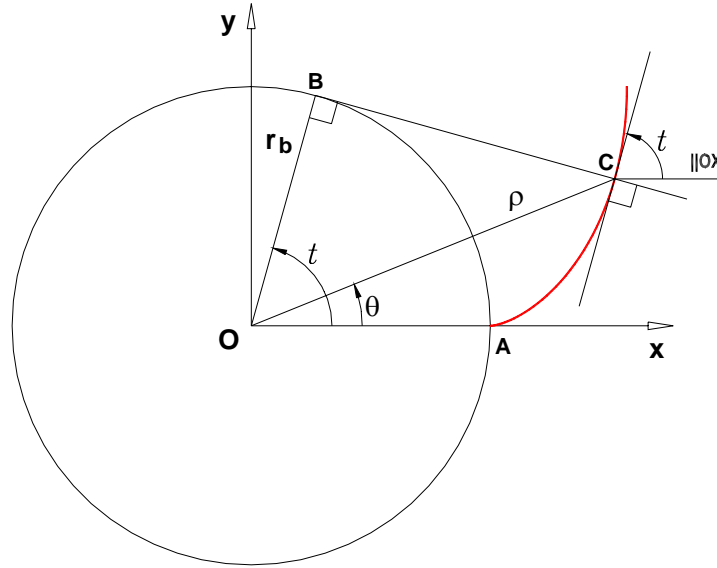


Fig. 1 The involute curve of a circle of radius  $r_b$

Because the string is inextensible, the length of the circular arch AB subtended by angle  $t$  is equal to length BC of the string:

$$BC = r_b \cdot t \quad (1)$$

Moreover, the center of curvature around point C of the involute is located right at the tangent point B between the string and the base circle, and has coordinates:

$$\begin{cases} x_B = r_b \cos(t) \\ y_B = r_b \sin(t) \end{cases} \quad (2)$$

In order to derive the equations of the involute, the following vector equation will be employed:

$$\vec{OC} = \vec{OB} - \vec{CB} \quad (3)$$

equivalent to the following scalar equations:

$$\begin{cases} x_C = r_b \cdot \cos(t) - r_b \cdot t \cdot \cos(t + \pi/2) \\ y_C = r_b \cdot \sin(t) - r_b \cdot t \cdot \sin(t + \pi/2) \end{cases} \quad (4)$$

Rearranging terms in equation (4) finally yields:

$$\begin{cases} x_C = r_b [\cos(t) + t \sin(t)] \\ y_C = r_b [\sin(t) - t \cos(t)] \end{cases} \quad (5)$$

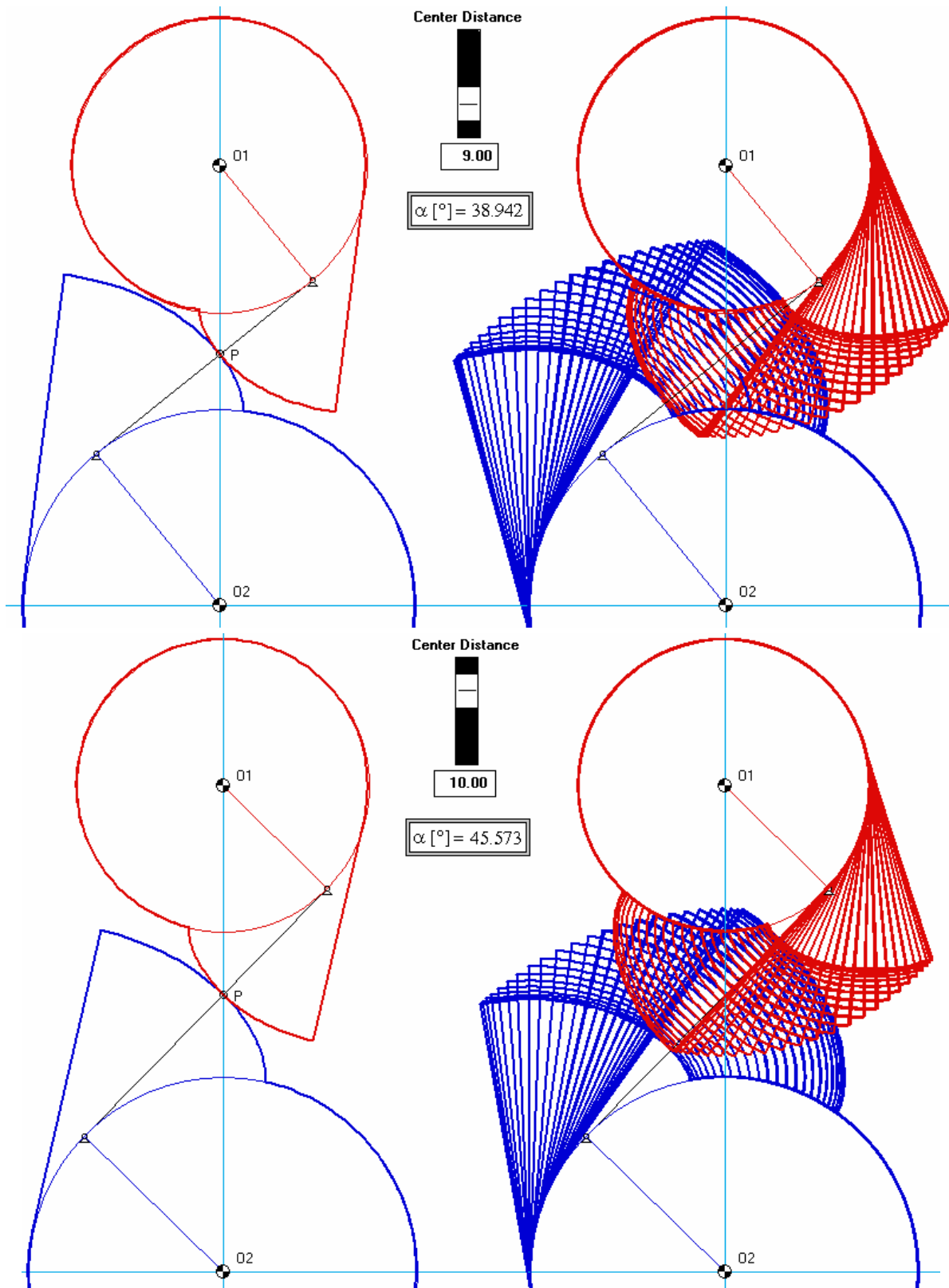


Fig. 2 WM 2D simulations showing that the transmission ratio between two involute profiles is insensitive to center distance modification.

Alternatively, the equations of the involute of a circle can be expressed using the polar angle  $\theta$ :

$$\theta = t - \arctan\left(\frac{BC}{OB}\right) = t - \arctan\left(\frac{r_b \cdot t}{r_b}\right) = t - \arctan(t) \quad (6)$$

and polar radius  $\rho$ :

$$\rho = \sqrt{OB^2 + BC^2} = r_b \sqrt{1 + t^2} \quad (7)$$

which yield the following alternative equations:

$$\begin{cases} x_c = \rho \cos(\theta) = r_b \sqrt{1 + t^2} \cdot \cos(t - \arctan(t)) \\ y_c = \rho \sin(\theta) = r_b \sqrt{1 + t^2} \cdot \sin(t - \arctan(t)) \end{cases} \quad (8)$$

### Involute Profile Mesh

Equations (8) were employed to generate inside WM 2D two polygonal bodies representing two involute curves attached to their base circles of radii  $r_{b1}=30$  mm and  $r_{b2}=40$  mm respectively (see Fig. 2). A *slider control* was defined and can be used to adjust the distance between the centers of these two base circles i.e. distance  $O_1O_2=a_w$  known as the *operating center distance*<sup>19</sup>. Two rotary motors were used to separately drive the two involutes, and were imposed correlated oscillatory motions of 0.4 and 0.3 radians amplitude respectively ensuing from the following angular velocities:

$$\begin{aligned} \omega_1 &= 0.4 \cos(t) \\ \omega_2 &= -\omega_1 r_{b1}/r_{b2} \end{aligned} \quad (9)$$

The initial positions between the two involutes (occurring for time  $t=0$ ) was set such that their contact point is collinear with centers  $O_1$  and  $O_2$ . This point, commonly noted P, is the *pitch point* of the two involutes<sup>19</sup>. Irrespective of the center distance value  $a_w$ , the following relations hold:

$$\begin{aligned} PO_1 &= a_w \frac{r_{b1}}{r_{b1} + r_{b2}} \\ PO_2 &= a_w \frac{r_{b2}}{r_{b1} + r_{b2}} \end{aligned} \quad (10)$$

where distances  $OP_1$  and  $OP_2$  are the pitch circle radii noted  $r_1$  and  $r_2$  respectively.

It is known that two involute gears in mesh can be replaced with two *friction wheels* of radii  $r_1$  and  $r_2$ , or with a *cross-belt transmission* (both assumed to have zero slip). In the latter case the two pulleys are the actual base circles while the belt is the common tangent to these circles. As the two involute profiles mesh, the contact point changes position, remaining however on the common tangent. According to *Arnhold-Kennedy Theorem of the Three Instant Centers*<sup>20</sup>, there will be pure rolling between the two profiles only when their contact point coincides with the pitch point P. Moreover, the farthest away from the pitch point P the contact occurs, the more sliding will occur between the two involute bodies, and therefore power losses are increased; in case of actual gear transmissions this implies that higher efficiencies can be obtained with smaller module teeth.

The crossbelt transmission equivalence provides an intuitive proof of the transmission ratio remaining constant as the two involute profiles mesh. This is a consequence of the *Fundamental Law of Tooth Gearing* being satisfied i.e. the common normal to the two involutes at the point of contact is the common tangent to the two base circles, and intersects the line of centers at a fixed point i.e. the pitch point P.

In order to maintain contact as the operating center distance is modified, the initial angles of the two involute-shaped polygons was programmed using the formula language of the software such that they will change the following amounts:

$$\Delta\alpha = \pm \left[ \arccos\left(\frac{r_{b1} + r_{b2}}{a_{ref}}\right) - \arccos\left(\frac{r_{b1} + r_{b2}}{a_w}\right) \right] \quad (11)$$

One can check that the first term in equation (11) is the reference pressure angle  $\alpha$  (see the next paragraph for details), and the second term is the pressure angle occurring for an arbitrary center distance  $a_w$ .

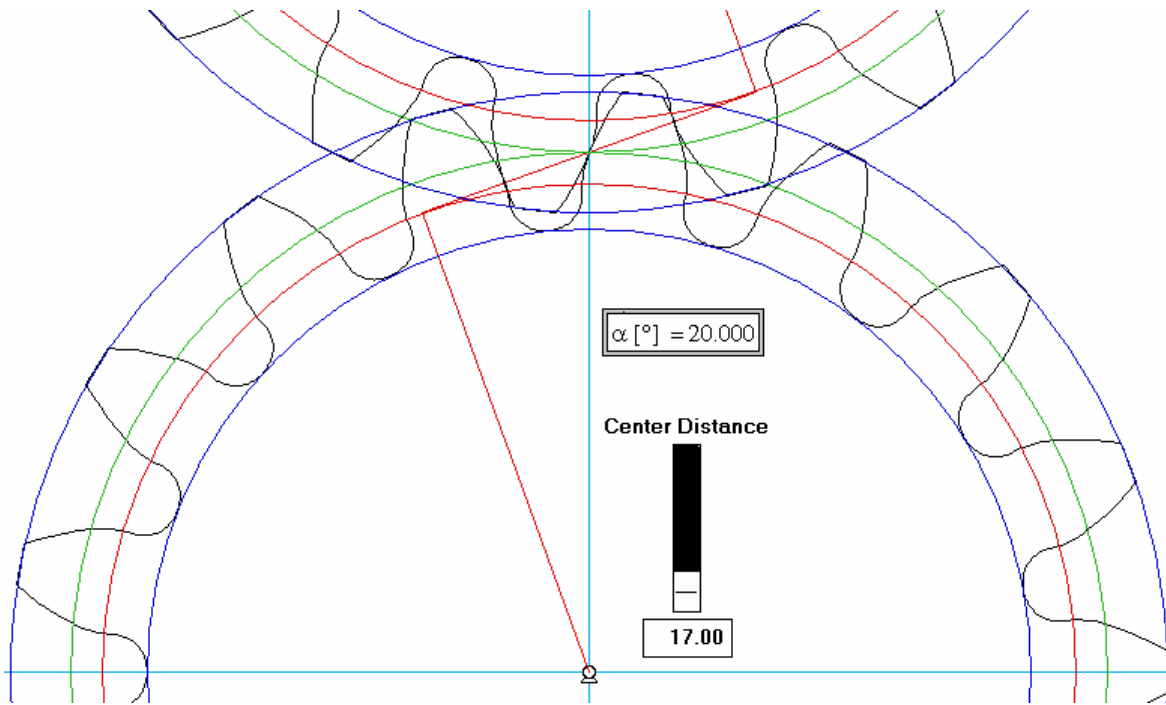


Fig. 3 WM 2D simulation of two full-depth-tooth standard gears, shown in reference center distance configuration, when the backlash is zero and the contact ratio is maximum.

### Involute External Gear Mesh

In order to demonstrate additional properties that involute gears have, a second WM 2D application was developed. It consists of two external *standard gears*\* having adjustable center distance as in the previous WM 2D application (Figures 3 and 4).

Since each gear consists of a number of identical involute curves equally spaced around the base circle, the concepts highlighted in the previous paragraph remain valid for any two involute gears

\*Standard gear is another term used for zero profile shift gears (see the **Involute Profile Generation** paragraph).

in mesh. Therefore, as the center distance increases, the pressure angle increases and vice versa, while the transmission ratio remains the same.

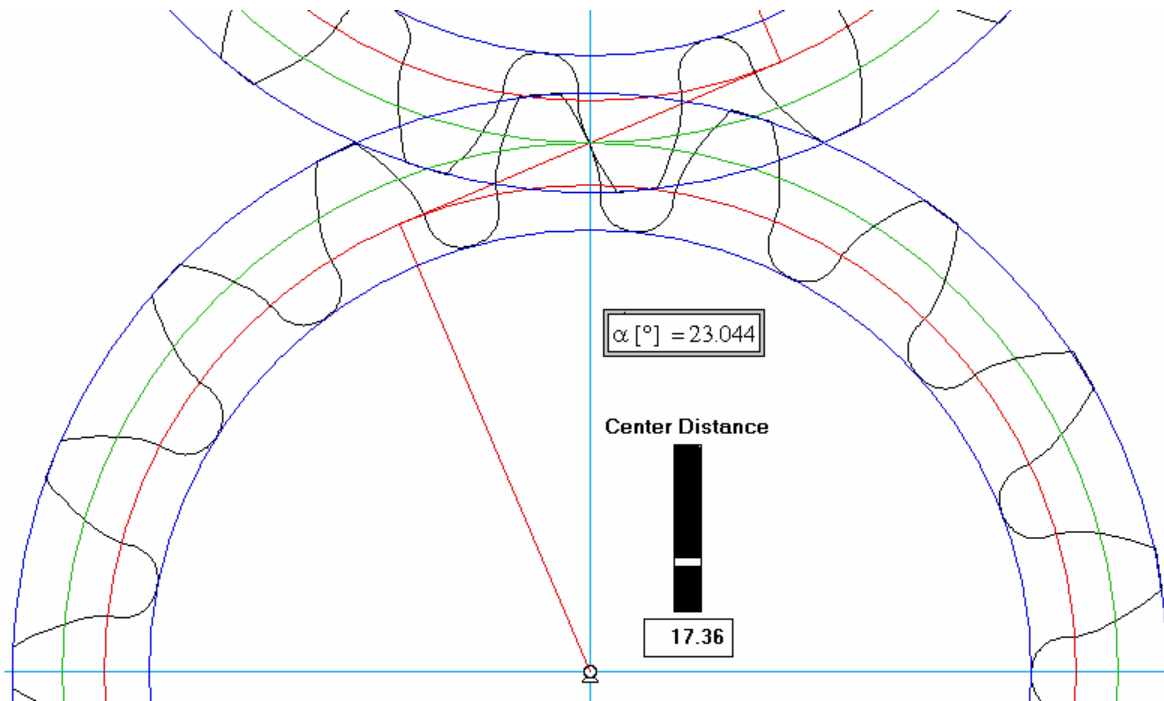


Fig. 4 WM 2D simulation of two full-depth-tooth standard gear pairs, shown in a configuration where the center distance is greater than the reference center distance, the backlash is nonzero and the contact ratio is diminished.

For those less familiar with the subject, the pressure angle between two smooth bodies is defined as the angle between the velocity vector of the contact point on the driven body, and the action force normal to the two bodies<sup>20</sup>. With this second WM 2D application, students have the possibility to explore how the pressure angle changes as the contact point between the two gears moves along the common tangent (aka *line of action*). In addition, the pressure angle changes value as the center distance is increased or decreased, and is also function of the direction in which the torque is transmitted (from gear 1 to gear 2 or vice versa). The only position in which the pressure angle is not dependent of which gear is the driving gear, is when the contact point coincides with the pitch point P (this is commonly referred to as the *pressure angle of two gears*, and is noted  $\alpha$  in the SI systems and  $\phi$  when US Customary units are used.). An important property of *standard gears* is that when the operating center distance equals the reference center distance ( $a_w = a_{ref}$ ), the pressure angle between the two gears is half the angle between the flanks of the teeth of the *basic rack*. The basic rack of a gear is obtained by hypothetically making the number of teeth of the respective gear equal infinitely<sup>19</sup>.

As Figure 3 shows, the minimum center distance is limited by the teeth of the two meshing gears making double contact (the *zero-backlash* gear pair arrangement). In case of standard external gears, the minimum center distance is:

$$a_{ref} = m(z_1 + z_2)/2 \quad (12)$$

where  $z_1$  and  $z_2$  are the number of teeth,  $m$  is the module.

Another important property of external gears the current WM 2D simulation reveals is that as the center distance is increased, the *contact ratio* is reduced (following the shortening of the *length of action*), while the *backlash* will of course increase (Figures 3 and 4). Finally, if the center distance is imposed a larger value that  $a_w$ , the approach to reducing the ensuing backlash is to employ *profile shift gears* as described next.

### **Involute Profile Generation as Envelopes of Rack and Gear Cutters**

In practice, involute gears are generated using a *rack cutter*, when a shaping motion (the actual cutting of the gear blank) alternates with the enveloping motion, where the reference line of the rack rolls without slip around the pitch circle of the gear. A second possibility is to envelope a *gear cutter*, the pitch diameter of which rolls without slip on the pitch circle of the blank. In both cases the gears obtained are said to be *zero profile shift* and their teeth are said to have *no correction*. When the cutter is displaced radially outwards from the zero shift position, a *positive shift profile* is obtained. Conversely, a radially inward displacement is called *negative profile shift*. The ratio between this displacement and the module of the gear is called *profile shift coefficient* and is commonly noted  $x$ <sup>19</sup>.

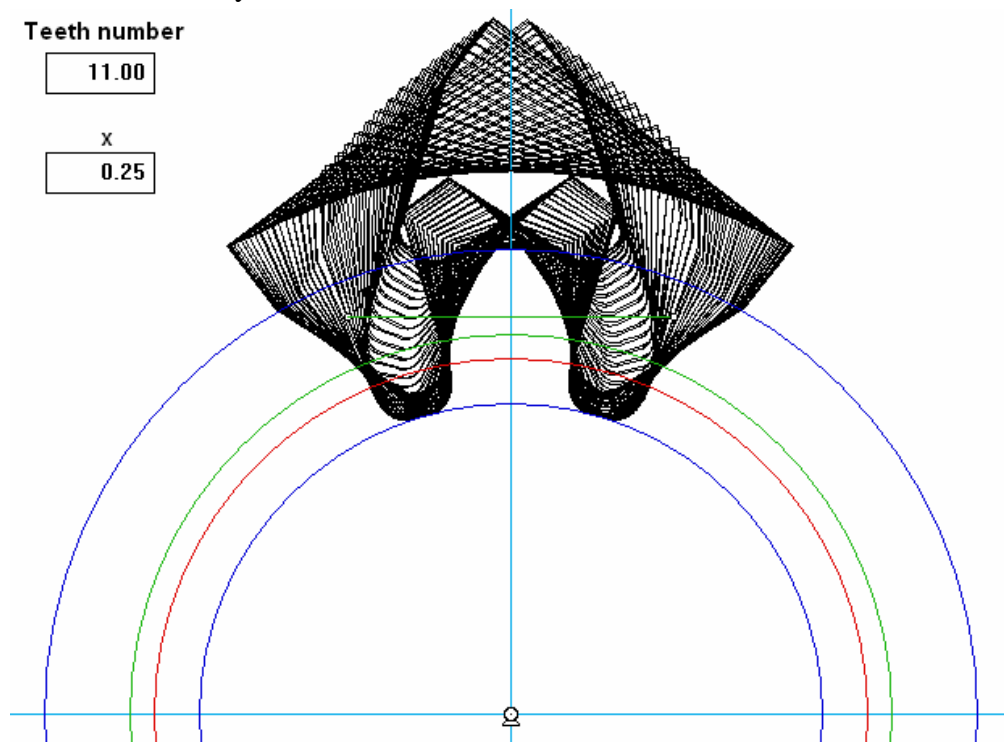


Fig. 5 WM 2D simulations of the gear cutting process using a rack tool

The above two involute-gear generation approaches were simulated using WM 2D software. With the rack-cutter application, a screenshot of which is given in Figure 5, students can generate as envelope one tooth of the desired gear. With the gear-cutter application (see Figures 6 and 7) an entire gear wheel (either internal or external) can be generated. Both WM 2D applications allow the user to select the number of teeth and profile shift coefficient and explore their individual or combined effect upon teeth geometry i.e. the teeth becoming undercut or pointed (see Figure 8).



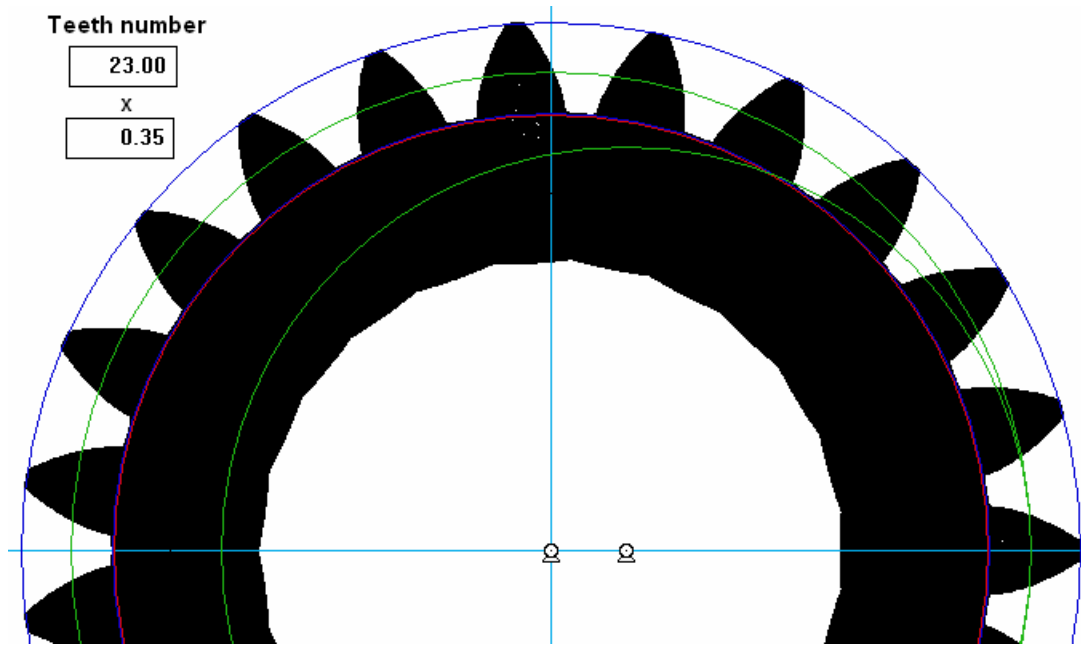


Fig. 6 WM 2D simulations of the gear-tool cutting an internal gear with 23 teeth and  $x=0.35$

Unfortunately, once these simulations have been performed within WM 2D, there is no convenient way of exporting the rack or gear cutter envelopes to a CAD package via the DXF format. This is because with current versions of the software, each animation frame has to be exported separately. Based on the author's own experience with the DXF format, it would be relatively easy to combine multiple frames in a single DXF file by placing each animation frame in a separate layer (although equally useful would be to have them accumulated in the same layer). Another disadvantage of the current DXF export strategy of the software is that all entities in a simulation are written to the DXF file, including those whose visibility was intentionally turned off.

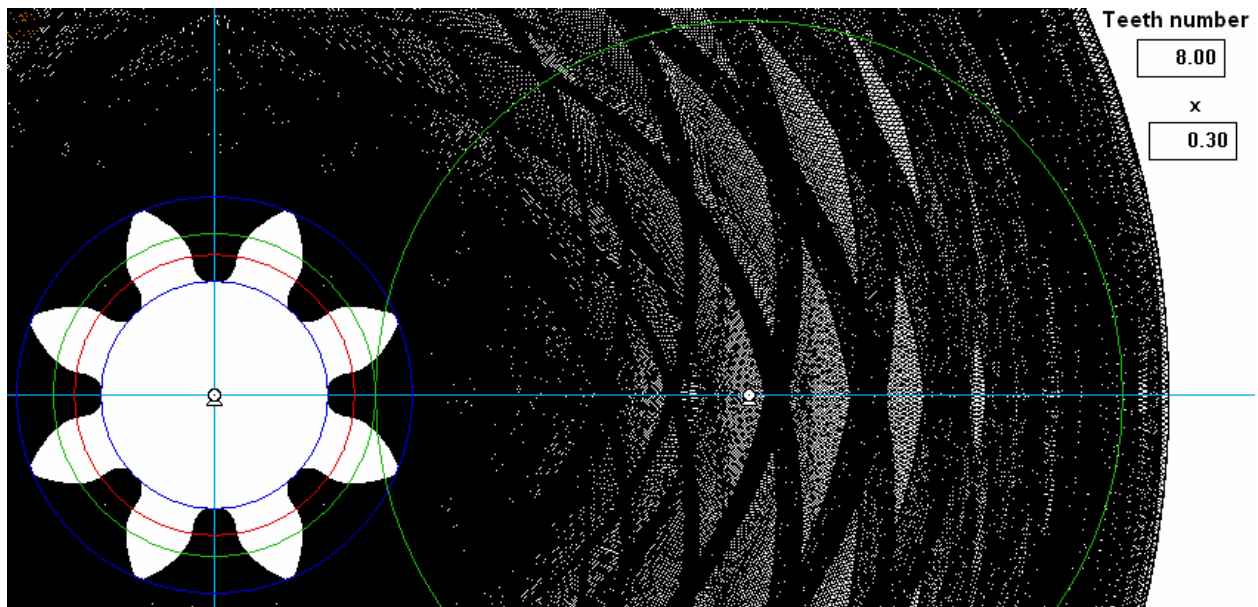


Fig. 7 WM 2D simulations of the gear-tool cutting an external gear with 8 teeth and  $x=0.30$

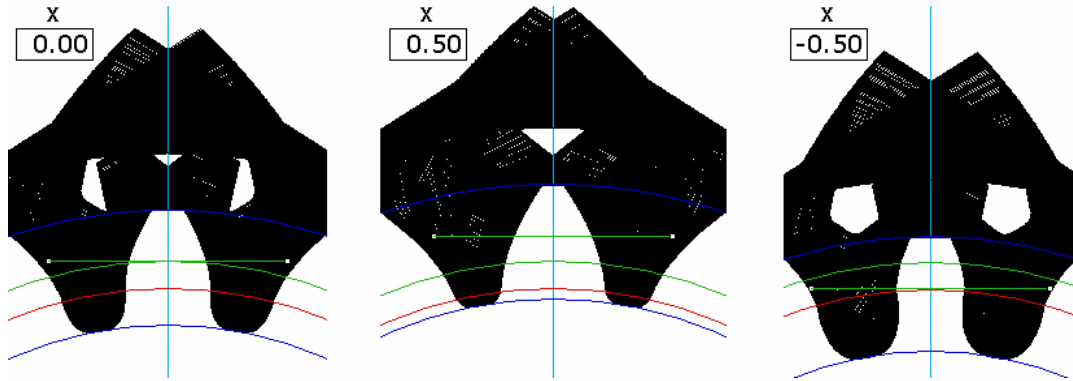


Fig. 8 Profile shift effects upon an external gear with 18 teeth (from left to right  $x = 0$ ,  $x = +0.5$  and  $x = -0.5$ )

### Planetary Gear Analysis

In comparison with fixed axle transmissions, in planetary gear trains there are gears (called *planet gears* or simply *planets*) the axes of which move on a circular path relative to at least two *central gears* sometimes called *sun gears*. The simplest of these transmissions have 2 degrees-of-freedom (DOF) and are known as *basic planetary gear trains*. According to Lévai<sup>21</sup> there are 12 such 2 DOF planetary gear trains, of the most common, two are shown in Figures 9 and 10.

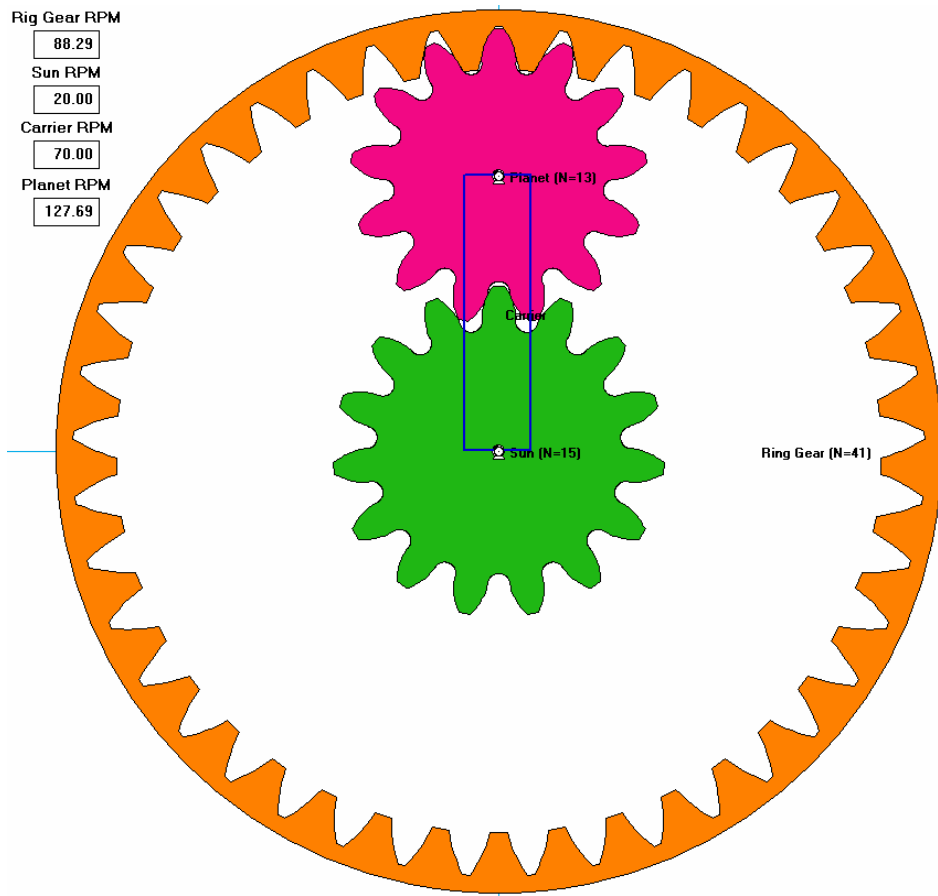


Fig. 9 WM 2D simulations of a basic planetary gear train comprising of one sun gear with 15 teeth, one ring gear with 41 teeth, planet carrier and a simple plane with 13 teeth.

The above mentioned Figures 9 and 10 are screenshots of two WM 2D simulations that allow students to verify if their kinematic calculations are correct i.e. if the rotational velocities of the central gears, planet gears and planet carrier correlate properly. Because of these gear trains have two degrees of freedom, the rotational speed of any two bodies must be specified (usually the central gears or one central gear and the planet carrier are given as inputs). If the speeds of the other two bodies are not correctly calculated (using the Tabular Method or Willis' Method of Motion Inversion), then once the simulation is performed, an “embarrassing” overlapping of gears can be observed.

Although the number of teeth of the gear wheels in these simulations is given, there are unlimited combinations of input rotational speeds that can be considered and students required to solve for the remaining unknown speeds.

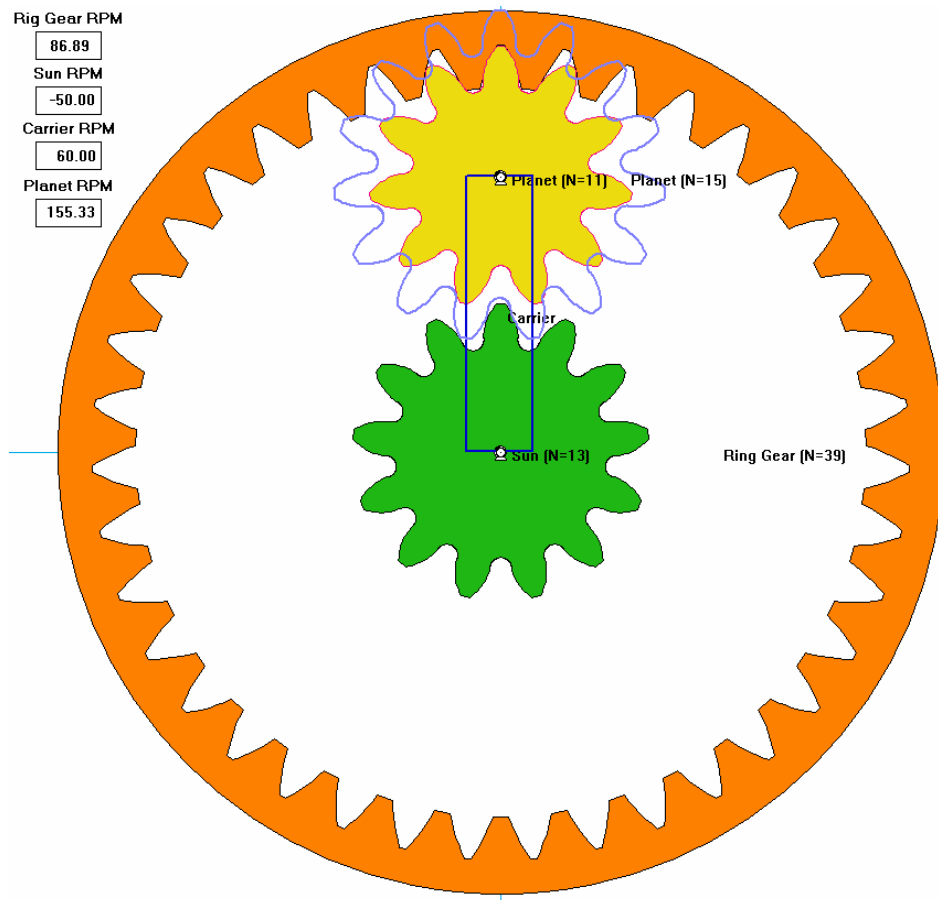


Fig. 10 WM 2D simulations of a basic planetary gear train comprising of one sun gear with 13 teeth, one ring gear with 39 teeth, planet carrier and a compound planet with 15 and 11 teeth respectively.

### Conclusions and Further Developments

A number of Working Model 2D applications have been presented that permit students to interactively explore the properties of involute gears and how these can be manufactured and used in gear transmissions.

Of these, two WM 2D applications illustrate that there is a kinematic equivalence between cross-belt transmissions and involute gears, and demonstrate how the center distance modification affects the pressure angle, backlash and contact ratio between the two gears.

Additional Working Model 2D applications illustrate how involute gears can be generated as envelopes using a rack or gear tool, and how the number of teeth and profile shift affects the teeth geometry (in particular undercut or the teeth becoming pointed).

Work is under way to generate more planetary gear simulation examples, including of the type used in automatic transmissions.

Since 2006 when these WM 2D simulations have been first introduced in the classroom, students received favorably both the demonstrations and homework problems formulated based on them.

In order to disseminate this positive experience, talks are under way to integrate these simulations as sample applications on the Working Model 2D installation CD, or on the companion CD of one of the Mechanisms textbooks listed in the Bibliography.

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