Chapter 8
Interval Estimation

- ▸ q Population Mean: σ Known
- ▸ q Population Mean: σ Unknown
- ▸ q Determining the Sample Size
- ▸ q Population Proportion

Margin of Error and the Interval Estimate

A point estimator cannot be expected to provide the exact value of the population parameter.

An <u>interval estimate</u> can be computed by adding and subtracting a <u>margin of error</u> to the point estimate.

Point Estimate +/− Margin of Error

The purpose of an interval estimate is to provide information about how close the point estimate is to the value of the parameter.

Margin of Error and the Interval Estimate

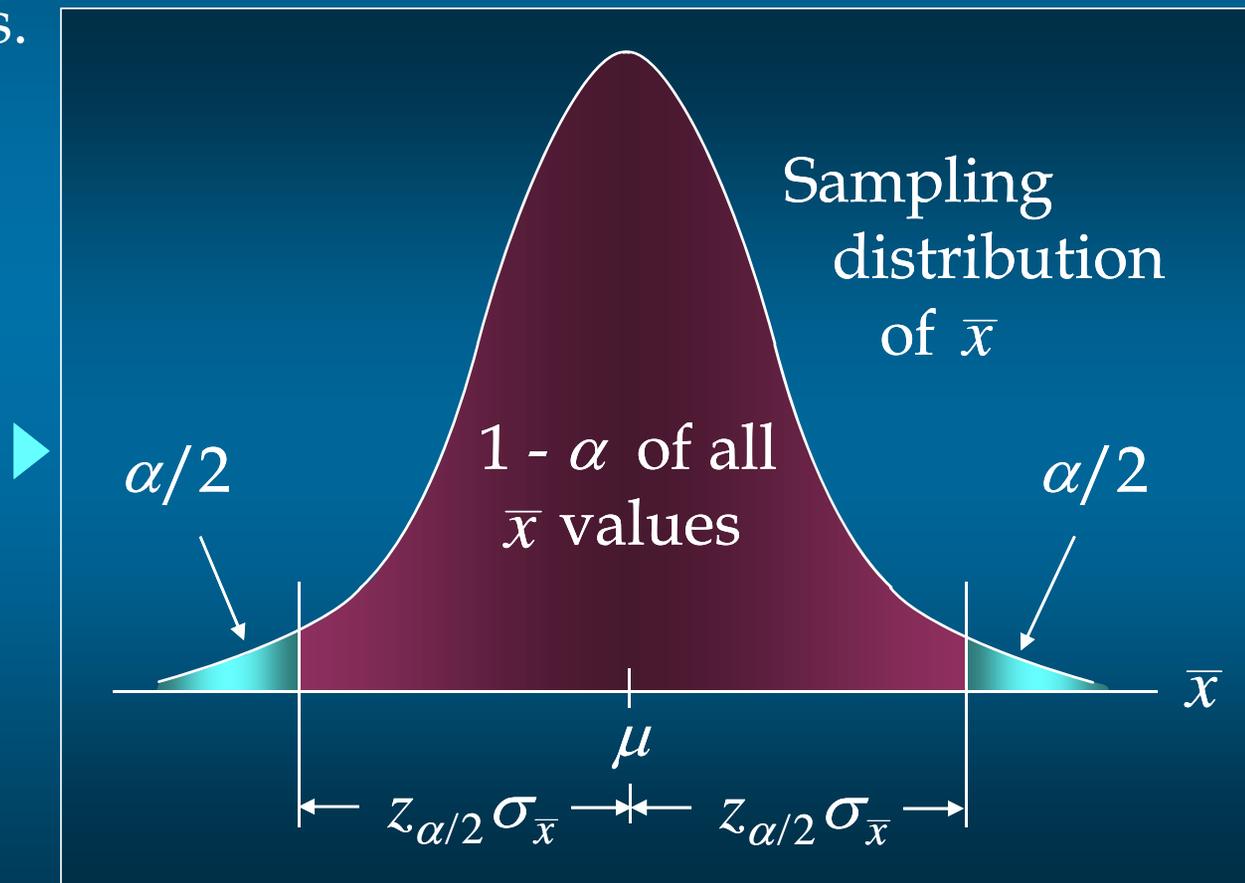The general form of an interval estimate of a population mean is

$$\bar{x} \pm \text{ Margin of Error}$$

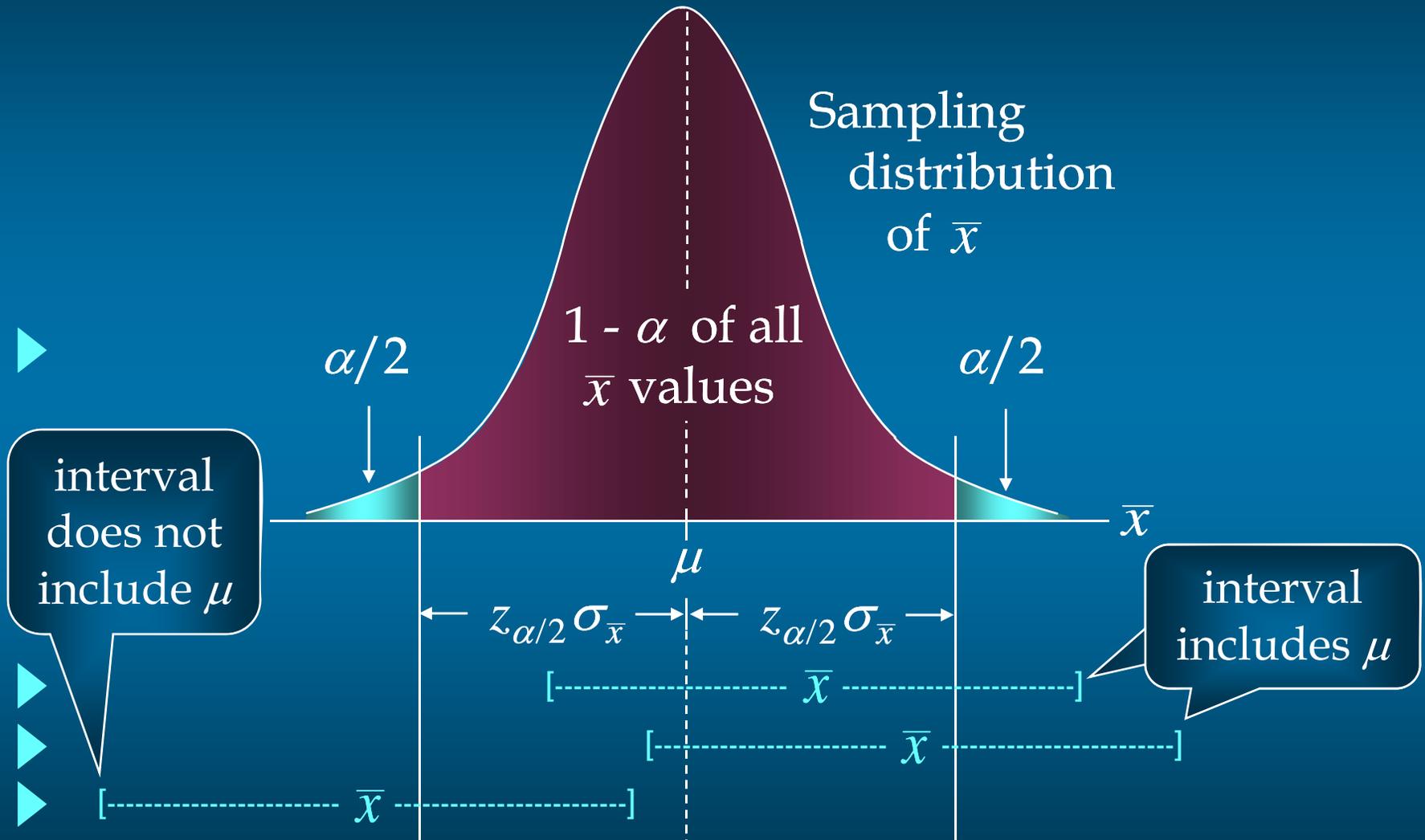# Interval Estimation of a Population Mean: σ Known

▶ q In order to develop an interval estimate of a population mean, the margin of error must be computed using either:

- the population standard deviation σ, or
- the sample standard deviation s

▶ q σ is rarely known exactly, but often a good estimate can be obtained based on historical data or other information.

▶ q We refer to such cases as the $\underline{\sigma\ known}$ case.

Interval Estimation of a Population Mean: σ Known

There is a $1 - \alpha$ probability that the value of a sample mean will provide a margin of error of $z_{\alpha/2}\sigma_{\bar{x}}$ or less.

Sampling distribution of $\bar{x}$

$\alpha/2$

$1 - \alpha$ of all $\bar{x}$ values

$\alpha/2$

μ

$\bar{x}$

$z_{\alpha/2}\sigma_{\bar{x}}$

$z_{\alpha/2}\sigma_{\bar{x}}$

Interval Estimate of a Population Mean: σ Known

q Interval Estimate of μ

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

where: $\bar{x}$ is the sample mean

$1 - \alpha$ is the confidence coefficient

$z_{\alpha/2}$ is the z value providing an area of $\alpha/2$ in the upper tail of the standard normal probability distribution

σ is the population standard deviation

n is the sample size

Interval Estimate of a Population Mean: σ Known

q Adequate Sample Size

▶ In most applications, a sample size of n = 30 is adequate.

▶ If the population distribution is highly skewed or contains outliers, a sample size of 50 or more is recommended.

Interval Estimate of Population Mean: σ Known

q Example: Discount Sounds

▶ Discount Sounds has 260 retail outlets throughout the United States. The firm is evaluating a potential location for a new outlet, based in part, on the mean annual income of the individuals in the marketing area of the new location.

▶ A sample of size n = 36 was taken; the sample mean income is $31,100. The population is not believed to be highly skewed. The population standard deviation is estimated to be $4,500, and the confidence coefficient to be used in the interval estimate is .95.

Interval Estimate of Population Mean: σ Known

▶ 95% of the sample means that can be observed are within $\pm\, 1.96\, \sigma_{\bar{x}}$ of the population mean μ.

▶ The margin of error is:

$$z_{\alpha/2} \frac{\sigma}{\sqrt{n}} = 1.96 \left(\frac{4{,}500}{\sqrt{36}} \right) = \boxed{1{,}470}$$

Thus, at 95% confidence, the margin of error is $1,470.

▶

Interval Estimate of Population Mean: σ Known

Interval estimate of μ is:

> $31,100 $\pm$ $1,470
>
> or
>
> $29,630 to $32,570

▶ We are <u>95% confident</u> that the interval contains the population mean.

Interval Estimation of a Population Mean: σ Unknown

▶ ⊓ If an estimate of the population standard deviation σ cannot be developed prior to sampling, we use the sample standard deviation s to estimate σ.

▶ ⊓ This is the <u>σ unknown</u> case.

▶ ⊓ In this case, the interval estimate for μ is based on the t distribution.

▶ ⊓ (We'll assume for now that the population is normally distributed.)

t Distribution

The *t* distribution is a family of similar probability distributions.

A specific *t* distribution depends on a parameter known as the degrees of freedom.

Degrees of freedom refer to the number of independent pieces of information that go into the computation of *s*.

t Distribution

A *t* distribution with more degrees of freedom has less dispersion.

As the number of degrees of freedom increases, the difference between the *t* distribution and the standard normal probability distribution becomes smaller and smaller.

t Distribution

Standard normal distribution

t distribution (20 degrees of freedom)

t distribution (10 degrees of freedom)

z, t

0
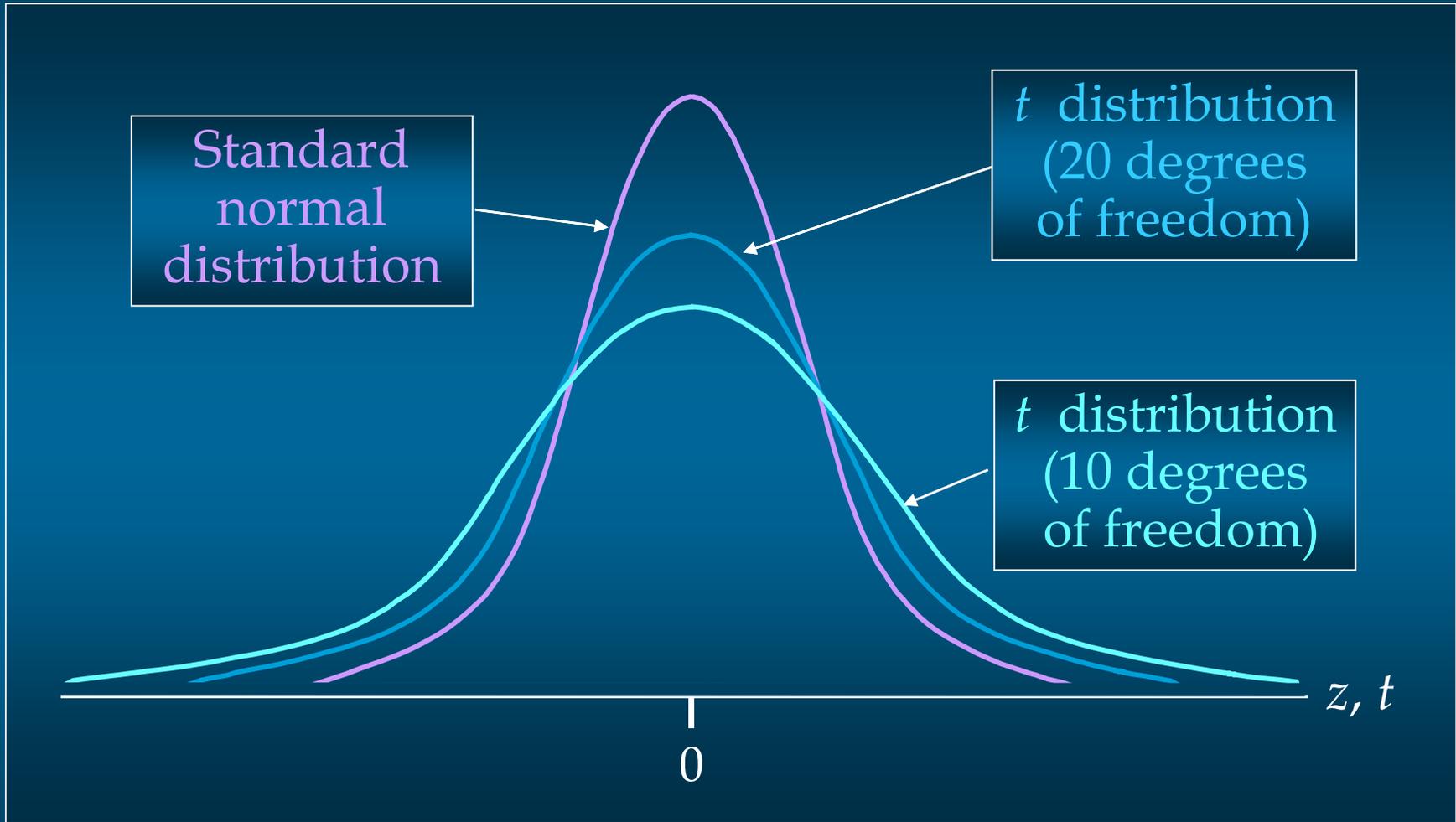
t Distribution

For more than 100 degrees of freedom, the standard normal *z* value provides a good approximation to the *t* value.

The standard normal *z* values can be found in the infinite degrees (∞) row of the *t* distribution table.

t Distribution

Degrees	Area in Upper Tail					
of Freedom	.20	.10	.05	.025	.01	.005
.	.	.	.	.	.	.
50	.849	1.299	1.676	2.009	2.403	2.678
60	.848	1.296	1.671	2.000	2.390	2.660
80	.846	1.292	1.664	1.990	2.374	2.639
100	.845	1.290	1.660	1.984	2.364	2.626
∞	.842	1.282	1.645	1.960	2.326	2.576

Standard normal
z values

Interval Estimation of a Population Mean: σ Unknown

q Interval Estimate
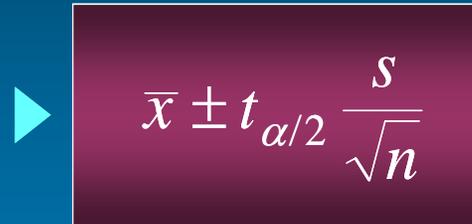
$$\bar{x} \pm t_{\alpha/2} \frac{s}{\sqrt{n}}$$

where: $1 - \alpha$ = the confidence coefficient

$t_{\alpha/2}$ = the t value providing an area of $\alpha/2$
in the upper tail of a t distribution
with $n - 1$ degrees of freedom

s = the sample standard deviation

Interval Estimation of a Population Mean: σ Unknown

q   Example: Apartment Rents

▶   A reporter for a student newspaper is writing an article on the cost of off-campus housing. A sample of 16 efficiency apartments within a half-mile of campus resulted in a sample mean of $650 per month and a sample standard deviation of $55.

Interval Estimation of a Population Mean: σ Unknown

q Example: Apartment Rents

▶   Let us provide a 95% confidence interval estimate of the mean rent per month for the population of efficiency apartments within a half-mile of campus. We will assume this population to be normally distributed.

Interval Estimation of a Population Mean: σ Unknown

▶ At 95% confidence, α = .05, and $\alpha/2$ = .025.

▶ $t_{.025}$ is based on $n - 1 = 16 - 1 = 15$ degrees of freedom.

▶ In the t distribution table we see that $t_{.025}$ = 2.131.

Degrees of Freedom	Area in Upper Tail					
	.20	.100	.050	.025	.010	.005
15	.866	1.341	1.753	2.131	2.602	2.947
16	.865	1.337	1.746	2.120	2.583	2.921
17	.863	1.333	1.740	2.110	2.567	2.898
18	.862	1.330	1.734	2.101	2.520	2.878
19	.861	1.328	1.729	2.093	2.539	2.861
.	.	.	.	.	.	.

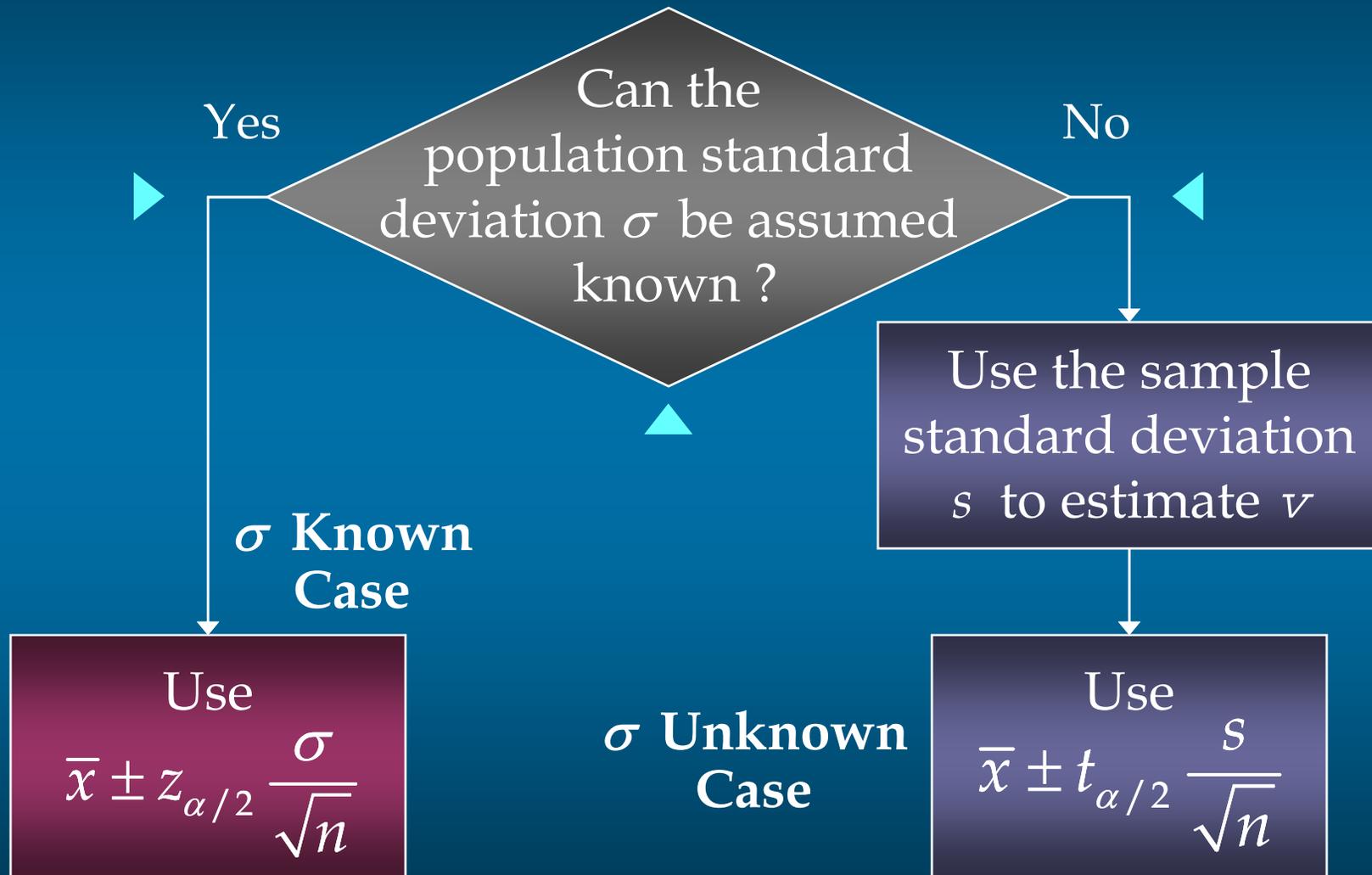# Interval Estimation of a Population Mean: σ Unknown

q Interval Estimate

$$\bar{x} \pm t_{.025}\frac{s}{\sqrt{n}}$$

$$650 \pm 2.131\frac{55}{\sqrt{16}} = 650 \pm 29.30$$

> We are 95% confident that the mean rent per month for the population of efficiency apartments within a half-mile of campus is between $620.70 and $679.30.

Summary of Interval Estimation Procedures
for a Population Mean

Can the population standard deviation σ be assumed known ?

Yes

No

σ **Known Case**

Use the sample standard deviation s to estimate v

Use
$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

σ **Unknown Case**

Use
$$\bar{x} \pm t_{\alpha/2} \frac{s}{\sqrt{n}}$$

Sample Size for an Interval Estimate of a Population Mean

Let E = the desired margin of error.

E is the amount added to and subtracted from the point estimate to obtain an interval estimate.

Sample Size for an Interval Estimate
of a Population Mean

▶ ❑ Margin of Error

$$E = z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

▶ ❑ Necessary Sample Size

$$n = \frac{(z_{\alpha/2})^2 \sigma^2}{E^2}$$

Sample Size for an Interval Estimate of a Population Mean

Recall that Discount Sounds is evaluating a potential location for a new retail outlet, based in part, on the mean annual income of the individuals in the marketing area of the new location.

Suppose that Discount Sounds' management team wants an estimate of the population mean such that there is a .95 probability that the sampling error is $500 or less.

How large a sample size is needed to meet the required precision?

Sample Size for an Interval Estimate of a Population Mean

► $$z_{\alpha/2}\,\frac{\sigma}{\sqrt{n}} = 500$$

► At 95% confidence, $z_{.025} = 1.96$. Recall that $\sigma = 4{,}500$.

$$n = \frac{(1.96)^2\,(4{,}500)^2}{(500)^2} = 311.17 = \boxed{312}$$

► A sample of size 312 is needed to reach a desired precision of $\pm$ \$500 at 95% confidence.