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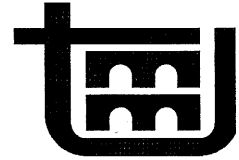


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STATIC EQUILIBRIUM DETERMINATION OF MULTI DOF SYSTEMS A REVIEW

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Abstract: Based on the actual case of a double pendulum connected to a linear spring, the main procedures for determining the static equilibrium of multi DOF systems are presented i.e. the method of solving the equilibrium equations of the system, applying the principle of virtual work, searching for the minimum of the total potential energy and the method of dynamic settling of the system. Among these, the last two are the most effective, the former one due to its simplicity and the latter due to its generality, which permits determining the equilibrium position of both conservative and non-conservative systems.

Keywords: multi-DOF systems, static equilibrium determination, iterative methods.

1. Introduction

Dynamic analysis of constrained multibody systems includes, among the time response and inverse dynamic (or kinetostatic) analysis the problem of determining the static equilibrium configuration.

Most of the research work carried out, starting with that of Livermore [3], concerns the determination of the equilibrium configuration of automobile chassis and suspension systems (which are complex spring-restrained mechanisms having a number of kinematic loops and multiple degrees of freedom) acted on by gravity loads and steady inertia forces [2], [3], [6]..[8].

Several methods for solving the equilibrium problem are known of which Haug [1] outlines those of *dynamic settling*, *solving the system of equilibrium equations* and *searching for the minimum of total potential energy*. A fourth major method is *that based on the principle of virtual work* and was considered by Visa et al [7], [8] and Talaba and Visa [6] for the determination the equilibrium position of front wheel suspension systems of automobiles relative to the body, a case in which the symmetry of these assemblies permits substantial simplification of the tedious analytical calculations involved.

The most universal method of finding the equilibrium configuration, no matter whether the applied forces are conservative or non-conservative, is to integrate the equation of motions until $\dot{q}_i = 0$ (where q_i are the generalised co-ordinates) with some tolerances. If necessary, artificial damping can be added to the system to accelerate convergence to an equilibrium state. In the case of dissipative non-conservative systems, for which many equilibrium configurations may exist, the dynamic settling approach is the only

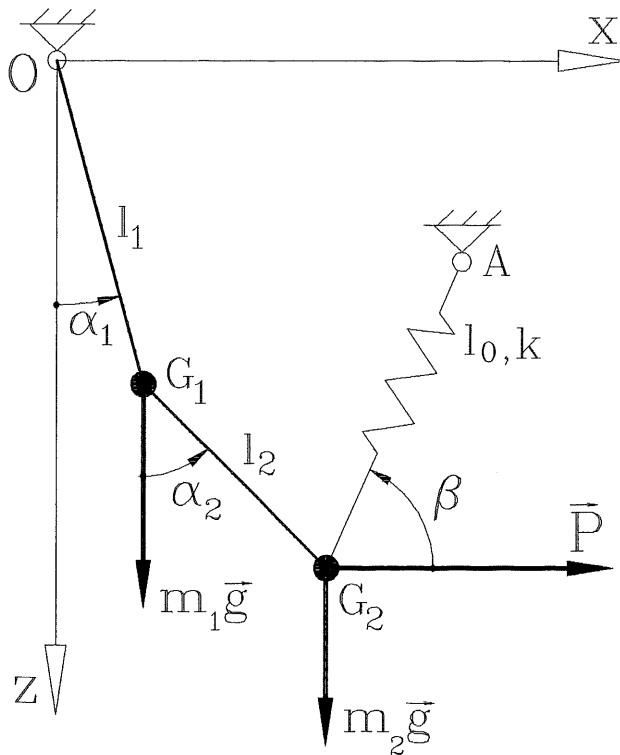


Fig. 1 A double pendulum with an elastic spring

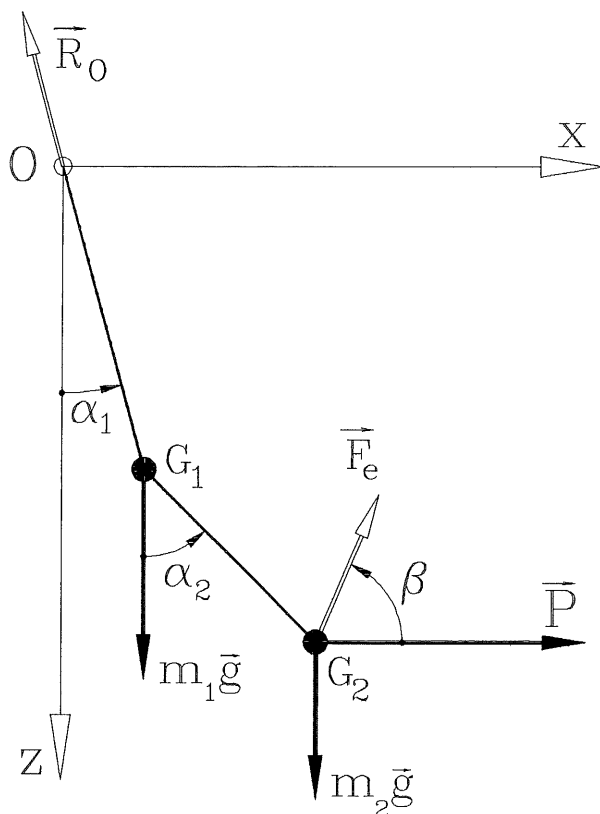


Fig. 2 The forces acting upon the system at the equilibrium state

valid method. However it is the most complicated and time consuming in computational terms. In the case of non-conservative systems the solutions found depend on the initial values chosen in the numerical integration subroutine.

Solving the equations of equilibrium by numerical methods, according to Haug, is not recommended, because of the possible convergence to an unwanted equilibrium configuration, which depends strongly on the initial estimates, including Lagrange's multipliers in the case of complicated nonholonomic constrained systems.

The application of the principle of virtual work requires the calculation of the velocities (or of the infinitesimal displacements) of the points where external forces act upon the system and therefore additional calculating effort.

In the present paper the above four methods are successively applied to determine the equilibrium configuration of a double pendulum with a linear spring having one end fixed to the frame and the other one attached to the outer lumped mass (see Fig. 1). This is not a real system, but serves to demonstrate the relative advantages of different approaches. The numerical values of the parameters of the system used in the analysis are: the lengths of the two members (assumed rigid) $l_1=0.45$ m and $l_2=0.35$ m, the masses of which $m_1=0.75$ kg and $m_2=0.5$ kg are considered lumped at the points G_1 and G_2 . The spring has the stiffness $k=9.5$ N/m and free length $l_0=0.2$ m. Following the example on page 213 in Thomson [5], a constant force P of a magnitude $P=1.8$ N acts horizontally.

2. Solving the equations of static equilibrium

In the most general case these equations of equilibrium of the system must be written for each body and solved together with the constraining equations describing the connections. For the simple case of the double pendulum in Fig. 1, the kinetostatics of the RRR dyad known from the mechanism theory can be applied. This permits the reaction forces occurring in the middle joint G_1 to be by-passed, their values being of no interest for the present purpose. Since the joint at O is assumed frictionless, the direction of the

reaction force R_0 is collinear with the axis of the element G_1O . For the whole dyad, the condition of equilibrium of the applied forces along the Ox and Oz axes are given by (see Fig. 2):

$$\begin{aligned} -R_0 \cdot \sin \alpha_1 + F_e \cdot \cos \beta + P &= 0 \\ -R_0 \cdot \cos \alpha_1 + F_e \cdot \sin \beta + m_1 \cdot g + m_2 \cdot g &= 0 \end{aligned} \quad (1)$$

and considering moments about G_1 :

$$l_2 \cdot \sin \alpha_2 \cdot (F_e \cdot \sin \beta - m_2 \cdot g) + l_2 \cdot \cos \alpha_2 \cdot (F_e \cdot \cos \beta + P) = 0 \quad (2)$$

Solving equations (1) and (2) numerically gives $\alpha_1=15.304^\circ$ and $\alpha_2=45.234^\circ$.

3. The method of virtual work

The above simplifications are seldom possible in most of the cases of multi DOF systems. Writing the static equilibrium equations is complicated by the necessity of accounting for all joints and constraint forces in free-body diagrams, whereas in the virtual work method these forces are excluded.

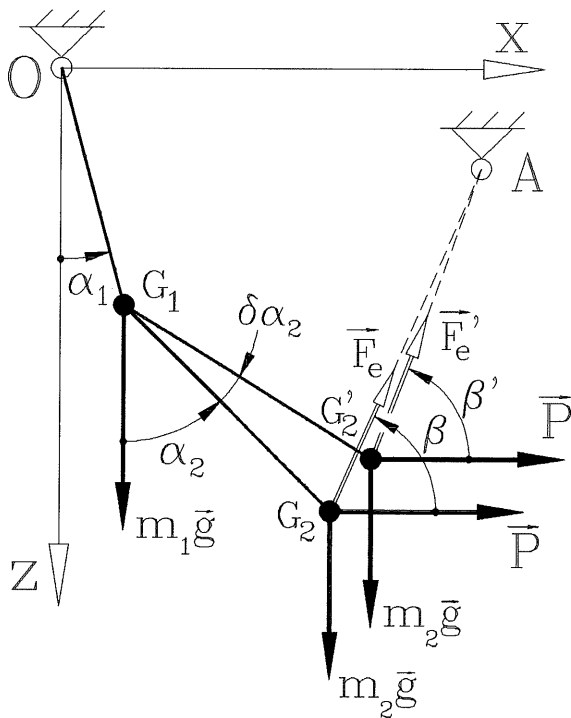


Fig. 3 Virtual displacement of co-ordinate α_2

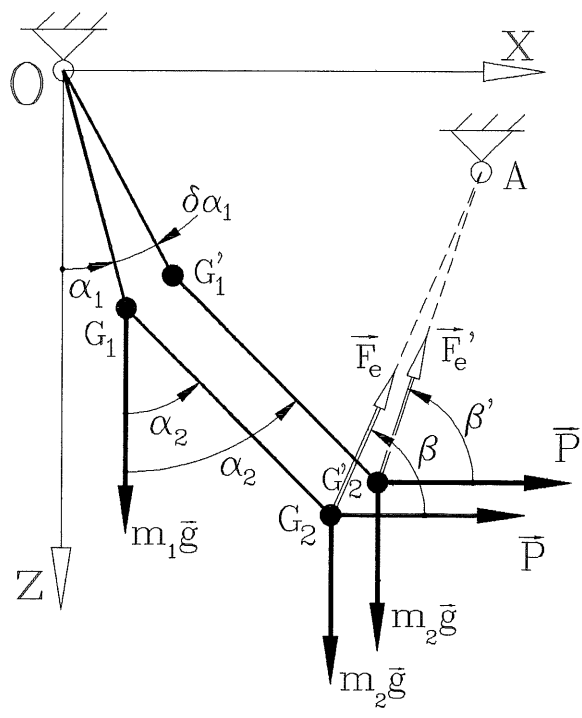


Fig. 4 Virtual displacement of co-ordinate α_1

The principle of virtual work can be stated as follows: *If a system in equilibrium under the action of a set of forces is given a virtual displacement, the virtual work done by the forces will be zero.* The analytical expression of the principle of virtual work can be summarised as:

$$\delta W = \sum_i \vec{F}_i \cdot \delta \vec{r}_i = 0, \quad (3)$$

where \vec{F}_i are the applied forces, (excluding all constraint forces and internal forces of frictionless joints) and $\delta \vec{r}_i$ are the virtual displacements. These virtual displacements are arbitrary variations of the co-ordinates irrespective to time, but compatible with the constraints of the system. Being an infinitesimal quantity, $\delta \vec{r}_i$ obeys all the rules of differential calculus. It differs from $d\vec{r}_i$ in that $d\vec{r}_i$ take place in time dt whereas $\delta \vec{r}_i$ is an arbitrary number assigned instantaneously irrespective of time.

The two virtual displacements $\delta \alpha_2$ and $\delta \alpha_1$ corresponding to the generalised co-ordinates α_2 and α_1 are schematically represented in Figs. 3 and 4. Because there is no significant change of geometry associated

with the virtual displacement, the forces acting on the system are assumed to remain unchanged for the calculation of δW and consequently \vec{F}_e and \vec{F}'_e in Figs. 3 and 4 are equal.

The virtual work due to $\delta\alpha_2$ is:

$$\delta W_2 = (m_2 \cdot g - F_e \cdot \sin\beta) \cdot \delta z_{G_2} + (P + F_e \cdot \cos\beta) \cdot \delta x_{G_2} \quad (4)$$

where the corresponding displacements of joint G_2 along the axes are:

$$\delta x_{G_2} = l_2 \cdot \cos\alpha_2 \cdot \delta\alpha_2$$

and (5)

$$\delta z_{G_2} = -l_2 \cdot \sin\alpha_2 \cdot \delta\alpha_2$$

Similarly for the infinitesimal displacement $\delta\alpha_1$ we have:

$$\delta W_1 = m_1 \cdot g \cdot \delta z_{G_1} + (m_2 \cdot g - F_e \cdot \sin\beta) \cdot \delta z_{G_2} + (P + F_e \cdot \cos\beta) \cdot \delta x_{G_2} \quad (6)$$

in this case δz_{G_1} , δx_{G_2} and δz_{G_2} being:

$$\delta x_{G_2} = l_1 \cdot \cos\alpha_1 \cdot \delta\alpha_1$$

and (7)

$$\delta z_{G_1} = \delta z_{G_2} = -l_1 \cdot \sin\alpha_1 \cdot \delta\alpha_1$$

The final system of equations becomes:

$$(F_e \cdot \sin\beta - m_2 \cdot g) \cdot \sin\alpha_2 + (P + F_e \cdot \cos\beta) \cdot \cos\alpha_2 = 0 \quad (8)$$

$$(F_e \cdot \sin\beta - m_2 \cdot g) \cdot \sin\alpha_1 - m_1 \cdot g \cdot \sin\alpha_1 + (P + F_e \cdot \cos\beta) \cdot \cos\alpha_1 = 0.$$

Equations (8) were solved numerically giving $\alpha_1=15.304^\circ$ and $\alpha_2=45.234^\circ$ which is the same in the previous case.

4. The minimum potential energy method

Dirichlet's principle gives the basis of searching for the configuration with minimum potential energy in obtaining the stable equilibrium position of the system. The condition of extrema (minimum or maximum) of the potential function $U(q_i)$ is expressed as:

$$\frac{\partial U(q_1 \dots q_n)}{\partial q_i} = 0, \quad (i = 1, n) \quad (9)$$

If it is difficult to select the proper set of generalised co-ordinates among the solutions of (9) the following supplementary conditions assuring that these co-ordinates to correspond to a minimum can be added:

$$\begin{bmatrix} a_{11} & \dots & a_{1k} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \dots & a_{kk} \end{bmatrix} > 0 \text{ with } a_{ij} = \frac{\partial^2 U(q_1 \dots q_n)}{\partial q_i \partial q_j} \quad (k = 1..n) \quad (10)$$

An alternative approach is to formulate a problem of optimisation and apply numerical searching algorithms. For gradient based minimisation methods it is necessary to calculate the derivatives of the total potential energy with respect to the generalised co-ordinates q_i . This might be sometimes convenient since the gradient of the total potential energy is the negative of the generalised applied force [1]. However, numerical calculation of the gradient is also feasible. It is much simpler to employ some optimisation procedures that do not require the calculation of any derivatives.

For the system under consideration, the total potential energy is a sum of: the gravity potentials of masses m_1 and m_2 , that due to the constant force P (which can also be considered as deriving from a potential) and of the elastic energy of the spring k . Choosing the reference positions in deriving the potentials of m_1 , m_2 and P is arbitrary. In this example the extreme positions of point G_2 along the positive axes Ox and Oz has been used as references giving:

$$U(\alpha_1, \alpha_2) = (l_1 - l_1 \cdot \cos\alpha_1) \cdot m_1 g + (l_1 + l_2 - l_1 \cdot \cos\alpha_1 - l_2 \cdot \cos\alpha_2) \cdot m_2 \cdot g + (l_1 + l_2 - l_1 \cdot \sin\alpha_1 - l_2 \cdot \sin\alpha_2) \cdot P + 1/2 \cdot k \cdot (AG_2 - l_0)^2 \quad (11)$$

where

$$AG_2 = \sqrt{(x_{G2} - x_A)^2 + (z_{G2} - z_A)^2} \quad (12)$$

and

$$x_{G2} = l_1 \cdot \sin \alpha_1 + l_2 \cdot \sin \alpha_2 \quad z_{G2} = l_1 \cdot \cos \alpha_1 + l_2 \cdot \cos \alpha_2 \quad (13)$$

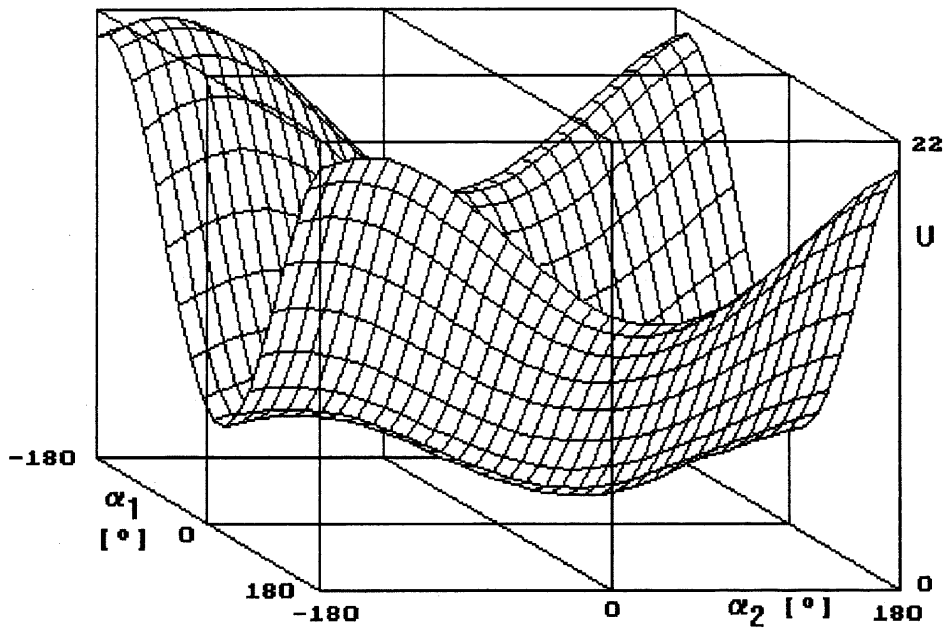


Fig. 5 3D plot of the potential function $U(\alpha_1, \alpha_2)$

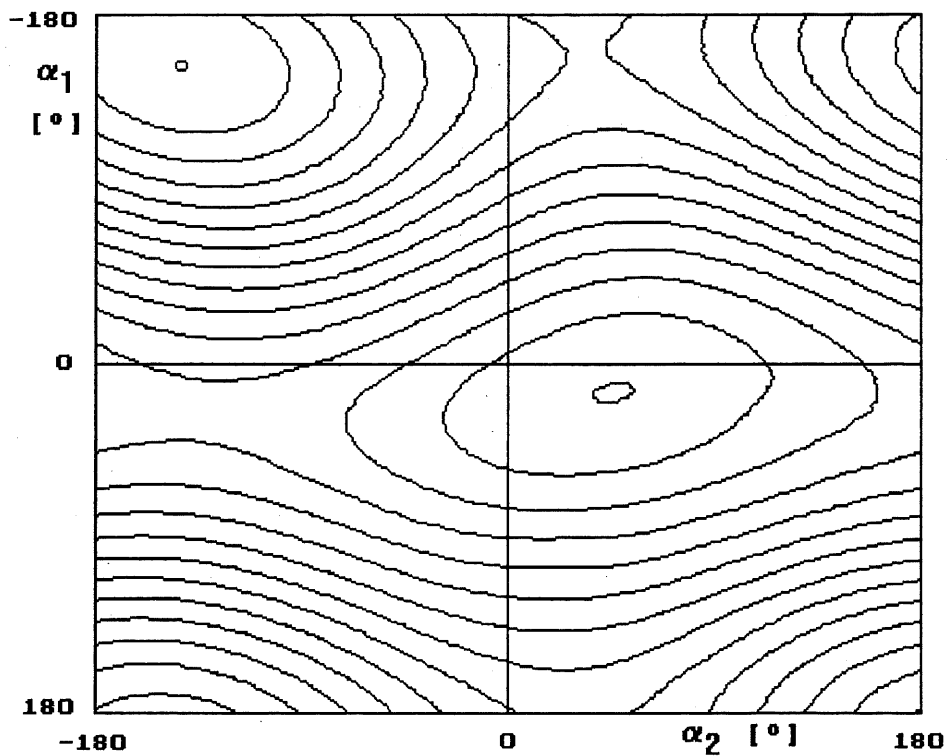


Fig. 6 Contour line plot of the potential function $U(\alpha_1, \alpha_2)$

A 3D plot of $U(\alpha_1, \alpha_2)$ is given in Fig. 5, and from the accompanying contour diagram in Fig. 6, one can approximate the position the two existing extrema of the potential function, of which the maximum corresponds to an unstable equilibrium position.

The necessary conditions of extrema (9) are in this case:

$$\begin{aligned} \frac{\partial U}{\partial \alpha_1} &= (m_1 + m_2) \cdot g \cdot l_1 \sin \alpha_1 - P \cdot l_1 \cos \alpha_1 + k \frac{AG_2 - l_0}{AG_2} l_1 \cdot \\ &[(x_{G2} - x_A) \cos \alpha_1 - (z_{G2} - z_A) \sin \alpha_1] = 0 \\ \frac{\partial U}{\partial \alpha_2} &= m_2 \cdot g \cdot l_2 \sin \alpha_2 - P \cdot l_2 \cos \alpha_2 + k \frac{AG_2 - l_0}{AG_2} l_2 \cdot \\ &[(x_{G2} - x_A) \cos \alpha_2 - (z_{G2} - z_A) \sin \alpha_2] = 0 \end{aligned} \quad (14)$$

The above system of equations can also be solved by applying an iterative procedure in an attempt to find the minimum of the potential function $U(\alpha_1, \alpha_2)$. The initial guesses can be chosen using an estimate of the position of the minimum as it appears in Fig. 6.

By using Neelder and Mead's simplex optimisation algorithm [4], the values of α_1 , and α_2 corresponding to the minimum of U have been determined. These values are the same as those obtained by numerically solving the system of equations (14), and identical with those obtained before by applying the principle of virtual work and solving the equilibrium equations.

5. Dynamic settling

The last method investigated is that of integrating the equations of motion until the system reaches an almost steady state.

The dynamic equations of the system have been derived using Lagrange's equations i.e.

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\alpha}_i} \right) - \frac{\partial T}{\partial \alpha_i} + \frac{\partial U}{\partial \alpha_i} = Q_i \quad (i = 1, 2) \quad (15)$$

where the kinetic energy T is:

$$T = \frac{1}{2} m_1 V_{G1}^2 + \frac{1}{2} m_2 V_{G2}^2 \quad (16)$$

where the squares of the velocities of points G_1 and G_2 are:

$$V_{G1}^2 = \dot{x}_{G1}^2 + \dot{y}_{G1}^2 = l_1^2 \cdot \dot{\alpha}_1^2 \quad (17)$$

$$V_{G2}^2 = \dot{x}_{G2}^2 + \dot{y}_{G2}^2 = l_1^2 \cdot \dot{\alpha}_1^2 + l_2^2 \cdot \dot{\alpha}_2^2 + 2 \cdot l_1^2 \cdot l_2^2 \cdot \cos(\alpha_1 - \alpha_2) \cdot \dot{\alpha}_1 \cdot \dot{\alpha}_2$$

Correspondingly, the partial derivatives of T appearing in (15) are as follows:

$$\frac{\partial T}{\partial \dot{\alpha}_1} = (m_1 + m_2) \cdot l_1^2 \cdot \dot{\alpha}_1 + m_2 \cdot l_1 \cdot l_2 \cdot \cos(\alpha_1 - \alpha_2) \cdot \dot{\alpha}_2 \quad (18)$$

$$\frac{\partial T}{\partial \dot{\alpha}_2} = m_2 \cdot l_2^2 \cdot \dot{\alpha}_2 + m_2 \cdot l_1 \cdot l_2 \cdot \cos(\alpha_1 - \alpha_2) \cdot \dot{\alpha}_1$$

so that:

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\alpha}_1} \right) &= (m_1 + m_2) \cdot l_1^2 \cdot \ddot{\alpha}_1 - m_2 \cdot l_1 \cdot l_2 \cdot \sin(\alpha_1 - \alpha_2) \cdot (\dot{\alpha}_1 - \dot{\alpha}_2) \cdot \dot{\alpha}_2 + \\ &+ m_2 \cdot l_1 \cdot l_2 \cdot \cos(\alpha_1 - \alpha_2) \cdot \ddot{\alpha}_2 \end{aligned} \quad (19)$$

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\alpha}_2} \right) &= m_2 \cdot l_2^2 \cdot \ddot{\alpha}_2 - m_2 \cdot l_1 \cdot l_2 \cdot \sin(\alpha_1 - \alpha_2) \cdot (\dot{\alpha}_1 - \dot{\alpha}_2) \cdot \dot{\alpha}_1 + \\ &+ m_2 \cdot l_1 \cdot l_2 \cdot \cos(\alpha_1 - \alpha_2) \cdot \ddot{\alpha}_1 \end{aligned}$$

and

$$\frac{\partial T}{\partial \alpha_1} = -m_2 \cdot l_1 \cdot l_2 \cdot \sin(\alpha_1 - \alpha_2) \cdot \dot{\alpha}_1 \cdot \dot{\alpha}_2$$

$$\frac{\partial T}{\partial \alpha_2} = m_2 \cdot l_1 \cdot l_2 \cdot \sin(\alpha_1 - \alpha_2) \cdot \dot{\alpha}_1 \cdot \dot{\alpha}_2$$
(20)

The generalised forces Q_i associated to the generalised co-ordinates α_1 and α_2 have been considered as the sum of a torsional viscous damping and a Coulomb damping occurring in joints G_1 and G_2 :

$$Q_1 = -\chi_1 \cdot \dot{\alpha}_1 - M_{G1} \cdot \text{sgn}(\dot{\alpha}_1) \quad \text{and} \quad Q_2 = -\chi_2 \cdot \dot{\alpha}_2 - M_{G2} \cdot \text{sgn}(\dot{\alpha}_2 - \dot{\alpha}_1)$$
(21)

where χ_1 and χ_2 are the damping coefficients. M_{G1} and M_{G2} are the friction torques which always oppose the relative rotation in the respective joints, and whose magnitudes may depend upon the reaction forces in the joints.

The final form of the system of non-linear equations considered for numerical integration is:

$$\ddot{\alpha}_1 = (b_1 \cdot c_2 - b_2 \cdot c_1) / (a_1 \cdot b_2 - a_2 \cdot b_1)$$

$$\ddot{\alpha}_2 = (a_1 \cdot c_2 - a_2 \cdot c_1) / (a_2 \cdot b_1 - a_1 \cdot b_2)$$
(22)

with

$$a_1 = (m_1 + m_2) \cdot l_2^2, \quad b_1 = m_2 \cdot l_1 \cdot l_2 \cdot \cos(\alpha_1 - \alpha_2),$$

$$c_1 = m_2 \cdot l_1 \cdot l_2 \cdot \sin(\alpha_1 - \alpha_2) \cdot \dot{\alpha}_2^2 + \frac{\partial U}{\partial \alpha_1} - Q_1,$$

$$a_2 = m_2 \cdot l_1 \cdot l_2 \cdot \cos(\alpha_1 - \alpha_2), \quad b_2 = m_2 \cdot l_2^2$$
(23)

and

$$c_2 = m_2 \cdot l_1 \cdot l_2 \cdot \sin(\alpha_2 - \alpha_1) \cdot \dot{\alpha}_1^2 + \frac{\partial U}{\partial \alpha_2} - Q_2$$

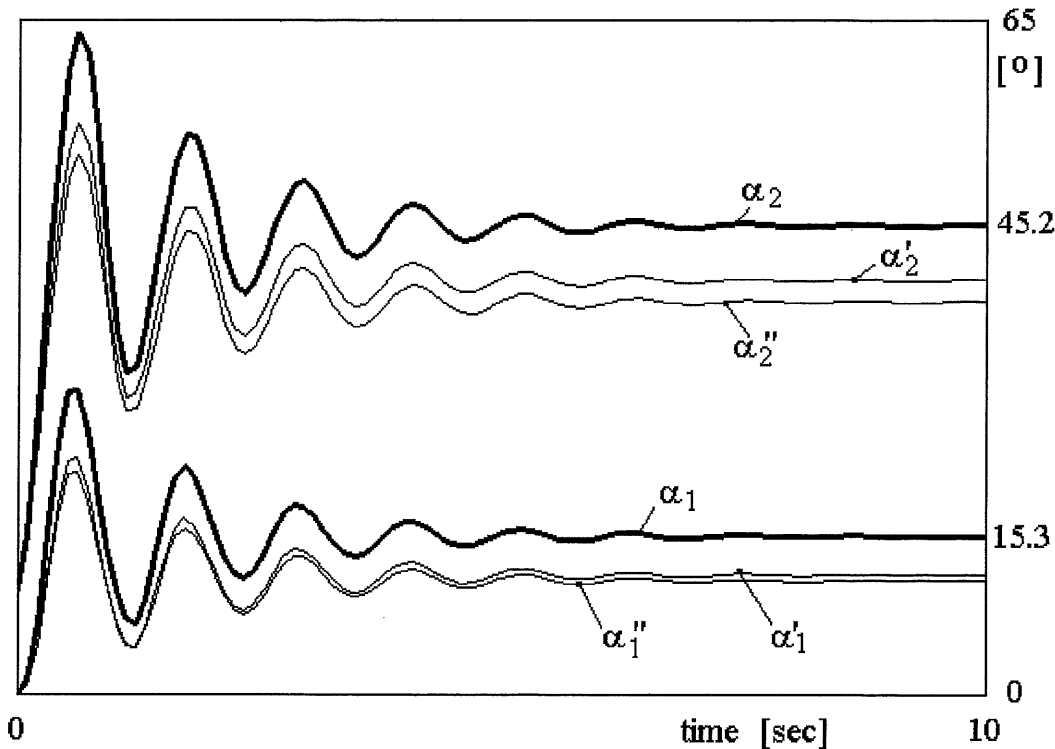


Fig. 7 Time response of the system (α_1 and α_2 with viscous damping only; $\alpha'_{1,2}$ and $\alpha''_{1,2}$ with both Coulomb and viscous damping)

By employing an adaptive stepsize fifth order Runge-Kutta subroutine [4], the time response of the system has been obtained.

The curves α_1 and α_2 in Fig. 7 show the response of the system with viscous damping only ($\chi_1=0.2$ N m s and $\chi_2=0.1$ N m s, $M_{G1}=0$ and $M_{G2}=0$). After 10 seconds, the values of α_1 and α_2 were found to be 15.318° and 45.259° respectively.

For comparison, other two cases shown by $\alpha'_{1,2}$ and $\alpha''_{1,2}$ have been considered with the same viscous damping but including Coulomb damping ($M'_{G1}=0.5$ N m and $M'_{G2}=0.4$ N m, and $M''_{G1}=0.6$ N m and $M''_{G2}=0.5$ N m respectively). The time variation plots of the co-ordinates α_1 and α_2 show, as expected, that the resting position varies for different M_{G1} and M_{G2} as compared with the system having viscous damping only.

6. Conclusions

It has been shown that among the possible approaches for determining the static equilibrium position of a multi DOF system, the method of searching for the minimum of the total potential energy is the simplest and most robust. The use of multi-start optimisation methods assures finding the global minimum of the potential function with certainty, this corresponding to a stable equilibrium configuration of the system.

The method of dynamic settling was the most time consuming but permitted determination of the equilibrium configuration of both conservative and non-conservative systems. It is known that for this type of dynamic system the equilibrium configuration is dependent on the initial values of the time dependent parameters.

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