all such elements to the same element in $\mathcal{Z}_{12}$. You can reverse the process of identification by looking at the inverse image of elements of $\mathcal{Z}_{12}$. The fact that this is always a coset of the kernel gives this particular process of identification an algebraic structure that can be usefully applied to other questions in group theory. In fact, the algebraic structure accounts for the fact that the set of elements identified with the unit element (i.e the kernel), entirely determines all other identifications (i.e., the cosets of the kernel).

In general, the effect of a homomorphism $h : G \to G'$ is to identify certain elements of G by mapping them to the same element of G'. The set of all things identified with a particular $x \in G$ is the coset Kx of x with respect to the normal subgroup K of all those things "identified" with the identity in G (mapped to the identity in G'). Moving from G to G' is a way of deciding (for a particular investigation) that elements which are similar in some sense will be treated as equal. Again, the reverse of this process is captured by the cosets of the kernel, K.

Another way of identifying things is the case of two different groups being considered "the same except for renaming", in which we identify each element of one group with a corresponding element of the other. This is captured by the concept of isomorphism which is a map that preserves everything related to group properties of any kind. An interesting consequence of the mathematics that does this is that the idea of preserving "everything related to group properties" is implemented by only three requirements. The map must be one-to-one, it must be onto, and it must preserve the operation, that is,

$$f(ab) = f(a)f(b).$$

Remark. One-to-one and kernels. There is a simple test for a homomorphism being one-to-one: the condition that the kernel be reduced to the identity (the smallest possible set since the identity will always be an element of the kernel). We leave the proof of this fact for the exercises, but it is interesting to observe that this condition can be interpreted as saying that there is no identification of the first kind. In other words, if we think of two elements being identified if they are mapped to the same element, then a map which is one-to-one will only identify an element with itself. In this sense, the two kinds of identification are mutually exclusive.

You investigated several examples of isomorphisms in the activities. For instance, in Activity 16 you were looking for isomorphisms between pairs of groups. Did you see that the two groups $\{1, -1\}$ and S_4/A_4 are isomorphic? Surely, two objects could not be more different than the group of two numbers 1,-1 (with the operation of ordinary multiplication) and the quotient group of the set of all cosets of A_4 in S_4 (with the operation of coset multiplication). Nevertheless, the fact that there is an isomorphism between them says that as groups, they are the same except for renaming. This means that you will never see a "group phenomenon" occur in one and not the other.

4.2.8 *EXERCISES*

1. Prove the rest of Proposition 4.4, p. 136: Let $f : G \to G'$ be a homomorphism. Then
   (a) For $a \in G$, $f(a^{-1}) = (f(a))^{-1}$.
   (b) If H is a subgroup of G then $f(H)$ is a subgroup of G'.
2. In the previous exercise, if we assume in addition that H is normal in G, does it follow that $f(H)$ is normal in G'? In $f(G)$?
3. Prove Proposition 4.5, p. 136 on invariants of onto homomorphisms. Show (by example) in each case that the "onto" assumption is indeed necessary.
4. Let G, G' be any two groups and define a map $t : G \to G'$ by $t(g) = e'$ for all $g \in G$. Show that t is a homomorphism. What is its kernel? (t is called a *trivial* homomorphism.)
5. Complete the proof that the map h defined beginning on p. 137 is a homomorphism of S_4 onto S_3.
6. Prove or find counterexamples for everything in Activities 13 and 16, p. 131.
7. Prove that all the "naive isomorphisms" we encountered in this book are indeed isomorphisms according to the formal definition given here.
8. Consider the map $x \mapsto \cos x + i \sin x$ (also denoted e^{ix}), $0 \leq x < 2\pi$. Show that this map is an embedding (i.e. one-to-one homomorphism) of the additive group $[[0, 2\pi), +_\pi]$ in the multiplicative group $[\mathcal{C} - \{0\}, *]$. What is its image? Can you think of some geometric meaning for this map?
9. Show that the map sending each complex number to its modulus (or absolute value) is a homomorphism of $[\mathcal{C} - \{0\}, *]$ onto the multiplicative group of positive real numbers. What is its kernel?
10. Show that the map $A \mapsto \det A$ is a homomorphism of the group of real $n \times n$-matrices with non-zero determinant onto the multiplicative group of non-zero real numbers. What is its kernel?
11. Prove that the additive groups $\mathcal{Z}$ and $\mathcal{Q}$ are not isomorphic.
12. Find non-trivial homomorphisms $\mathcal{Z}_3 \to \mathcal{Z}_{15}$ and $\mathcal{Z}_{15} \to \mathcal{Z}_3$.
13. Find non-trivial homomorphisms $S_3 \to \mathcal{Z}_{12}$ and $\mathcal{Z}_{12} \to S_3$.